

Quantum Field Theory

Set 12: solutions

Exercise 1

- We have

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} [J_{\nu\rho}, P_\sigma] + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\sigma J_{\nu\rho} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} [J_{\nu\rho}, P_\sigma] + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma}, \quad (1)$$

where in the last equality we relabeled $\sigma \leftrightarrow \nu$ and used anti-symmetry of $\epsilon^{\mu\nu\rho\sigma}$ and $J_{\rho\sigma}$.

Now all we have to show is that the term involving the commutator is zero. It follows from the algebra (eq. (3.191) in the lecture notes),

$$[J_{\nu\rho}, P_\sigma] = i(\eta_{\sigma\rho} P_\nu - \eta_{\sigma\nu} P_\rho), \quad (2)$$

that

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} [J_{\nu\rho}, P_\sigma] = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} (\eta_{\sigma\rho} P_\nu - \eta_{\sigma\nu} P_\rho) = 0. \quad (3)$$

The first term is zero because $\eta_{\sigma\rho}$ is symmetric and its indices are contracted with $\epsilon^{\mu\nu\rho\sigma}$ which is anti-symmetric. The second term is zero for the same reason.

- 1. It follows from the algebra (eq. (3.192) from the lectures notes) and anti-symmetry of the Levi-Civita tensor that

$$W^\mu P_\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma P_\mu = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} [P_\sigma, P_\mu] = 0. \quad (4)$$

- 2. We have

$$[P^\mu, W^\nu] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma} [P^\mu, J_{\alpha\beta} P_\sigma] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma} ([P^\mu, J_{\alpha\beta}] P_\sigma + J_{\alpha\beta} [P^\mu, P_\sigma]). \quad (5)$$

Now we make use of the algebra, in particular $[P^\mu, P_\sigma] = 0$ and eq. (2) to find

$$[P^\mu, W^\nu] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma} ([P^\mu, J_{\alpha\beta}] P_\sigma) = \frac{i}{2}\epsilon^{\nu\alpha\rho\sigma} (\delta_\alpha^\mu P_\rho - \delta_\rho^\mu P_\alpha) P_\sigma = 0, \quad (6)$$

due to antisymmetry of the Levi-Civita tensor.

- 3. We have

$$[J^{\mu\nu}, W^\rho] = \frac{1}{2}\epsilon^{\rho\alpha\beta\sigma} [J^{\mu\nu}, J_{\alpha\beta} P_\sigma] = \frac{1}{2}\epsilon^{\rho\alpha\beta\sigma} ([J^{\mu\nu}, J_{\alpha\beta}] P_\sigma + J_{\alpha\beta} [J^{\mu\nu}, P_\sigma]). \quad (7)$$

We again make use of the algebra,

$$[J^{\mu\nu}, J_{\alpha\beta}] = i(\delta_\alpha^\nu J_\beta^\mu - \delta_\alpha^\mu J_\beta^\nu + \delta_\beta^\mu J_\alpha^\nu - \delta_\beta^\nu J_\alpha^\mu), \quad [J^{\mu\nu}, P_\sigma] = i(\delta_\sigma^\nu P^\mu - \delta_\sigma^\mu P^\nu), \quad (8)$$

to get

$$[J^{\mu\nu}, W^\rho] = \frac{i}{2}(\epsilon^{\rho\nu\beta\sigma} J_\beta^\mu P_\sigma + \epsilon^{\rho\alpha\nu\sigma} J_\alpha^\mu P_\sigma + \epsilon^{\rho\alpha\beta\nu} J_{\alpha\beta} P_\mu) - (\mu \leftrightarrow \nu). \quad (9)$$

We now factor out $J_{\alpha\beta} P_\sigma$ to have

$$[J^{\mu\nu}, W^\rho] = \frac{i}{2} J_{\alpha\beta} P_\sigma (\eta^{\mu\alpha} \epsilon^{\rho\nu\beta\sigma} + \eta^{\mu\beta} \epsilon^{\rho\alpha\nu\sigma} + \eta^{\mu\sigma} \epsilon^{\rho\alpha\beta\nu}) - (\mu \leftrightarrow \nu). \quad (10)$$

We now make use of the identity¹

$$\eta^{\mu\nu} \epsilon^{\rho\alpha\beta\sigma} = \eta^{\mu\rho} \epsilon^{\nu\alpha\beta\sigma} + \eta^{\mu\alpha} \epsilon^{\rho\nu\beta\sigma} + \eta^{\mu\beta} \epsilon^{\rho\alpha\nu\sigma} + \eta^{\mu\sigma} \epsilon^{\rho\alpha\beta\nu} \quad (12)$$

¹This identity follows from expanding the determinant relation

$$\det(A) \epsilon^{\rho\alpha\beta\sigma} = A_{\rho'}^\rho A_{\alpha'}^\alpha A_{\beta'}^\beta A_{\sigma'}^\sigma \epsilon^{\rho'\alpha'\beta'\sigma'} \quad (11)$$

to linear order in $\omega_{\rho'}^\rho$, where $A_{\rho'}^\rho = \delta_{\rho'}^\rho + \omega_{\rho'}^\rho + O(\omega^2)$. The left-hand-side becomes proportional to the trace $\text{tr } \omega = \eta^{\mu\nu} \omega_{\mu\nu}$, matching to the left-hand-side of (12). Correspondingly, the right-hand-sides will match and given that $\omega^{\mu\nu}$ is arbitrary, the identity (12) will hold.

to have

$$[J^{\mu\nu}, W^\rho] = \frac{i}{2} J_{\alpha\beta} P_\sigma (\eta^{\mu\nu} \epsilon^{\rho\alpha\beta\sigma} - \eta^{\mu\rho} \epsilon^{\nu\alpha\beta\sigma}) - (\mu \leftrightarrow \nu) = \quad (13)$$

$$= -\frac{i}{2} J_{\alpha\beta} P_\sigma \eta^{\mu\rho} \epsilon^{\nu\alpha\beta\sigma} + \frac{i}{2} J_{\alpha\beta} P_\sigma \eta^{\nu\rho} \epsilon^{\mu\alpha\beta\sigma} = i(\eta^{\nu\rho} W^\mu - \eta^{\mu\rho} W^\nu). \quad (14)$$

Note that replacing W^ρ with the momentum P^ρ in the above yields the correct commutation relation between $J^{\alpha\beta}$ and P^ρ . This result is a consequence of the fact that W^ρ transforms as a 4-vector under Lorentz transformations (just like P^ρ).

4. We have

$$[W^\mu, W^\nu] = \frac{1}{2} \epsilon^{\nu\alpha\rho\sigma} [W^\mu, J_{\alpha\beta} P_\sigma] = \frac{1}{2} \epsilon^{\nu\alpha\rho\sigma} ([W^\mu, J_{\alpha\beta}] P_\sigma + J_{\alpha\beta} [W^\mu, P_\sigma]). \quad (15)$$

We now make use of the results found in points 2 and 3. We find

$$[W^\mu, W^\nu] = \frac{i}{2} \epsilon^{\nu\alpha\rho\sigma} (\delta_\alpha^\mu W_\rho - \delta_\rho^\mu W_\alpha) P_\sigma = i \epsilon^{\nu\alpha\rho\sigma} W_\rho P_\sigma. \quad (16)$$

- To show that W^2 is a Casimir we have to show that it commutes with the Poincare generators P^μ and $J^{\mu\nu}$. We have

$$[W^2, P^\mu] = W_\alpha [W^\alpha, P^\mu] + [W^\alpha, P^\mu] W_\alpha = 0 \quad (17)$$

due to the result derived in point 2.

We also have

$$[W^2, J^{\mu\nu}] = W_\alpha [W^\alpha, J^{\mu\nu}] + [W^\alpha, J^{\mu\nu}] W_\alpha = i[W^\mu, W^\nu] + i[W^\nu, W^\mu] = 0, \quad (18)$$

where we made use of the result derived in point 3.

Exercise 2

- At first, let us simply write the infinitesimal transformation for a Lorentz vector in terms of θ^k and η^k .

$$v'^\mu = \Lambda^\mu_\nu v^\nu \simeq (\delta^\mu_\nu + \omega^\mu_\nu) v^\nu = (\delta^\mu_\nu + g_{\nu\alpha} \omega^{\mu\alpha}) v^\nu \quad (19)$$

$$v'^0 = v^0 + g_{\nu\alpha} \omega^{0\alpha} v^\nu = v^0 + g_{\nu i} \omega^{0i} v^\nu = v^0 - \delta_{ji} \omega^{0i} v^j = v^0 + \eta^i v^i \quad (20)$$

$$v'^i = v^i + g_{\nu\alpha} \omega^{i\alpha} v^\nu = v^i + g_{\nu j} \omega^{ij} v^\nu + g_{\nu 0} \omega^{i0} v^\nu = v^i + g_{kj} \omega^{ij} v^k + g_{00} \omega^{i0} v^0 \\ = v^i + g_{kj} \epsilon^{ijl} \theta^l v^k + g_{00} \eta^i v^0 = v^i - \epsilon^{ijl} \theta^l v^j + \eta^i v^0$$

where $g_{\mu\nu}$ is the Minkowski metric.

- Now, we want to show:

$$\Lambda_L(\theta, \eta) v^\mu \sigma_\mu \Lambda_L^\dagger(\theta, \eta) = \Lambda^\mu_\nu(\theta, \eta) v^\nu \sigma_\mu \quad (21)$$

Let us call $v \equiv v^\mu \sigma_\mu$. Expanding the left hand side to linear order:

$$v'^\mu \sigma_\mu = \exp[-(\vec{\eta} + i\vec{\theta}) \cdot \vec{\sigma}/2] v \exp[-(\vec{\eta} - i\vec{\theta}) \cdot \vec{\sigma}/2] \simeq [1 - (\vec{\eta} + i\vec{\theta}) \cdot \vec{\sigma}/2] v [1 - (\vec{\eta} - i\vec{\theta}) \cdot \vec{\sigma}/2] \\ = v - \frac{\eta^i}{2} \{\sigma^i, v\} - \frac{i\theta^i}{2} [\sigma^i, v] = v - \frac{\eta^i}{2} (2v^0 \sigma^i - 2v^i \sigma^0) + \frac{i\theta^i}{2} (2i\epsilon^{ijk} v^j \sigma^k) \\ = v + \eta^i v^i \sigma_0 + (+\epsilon^{ijl} \theta^l v^j + \eta^i v_0) \sigma_i.$$

which, by equating the components of σ_μ on both sides of the equation, corresponds exactly to the transformations (19), (20).

- Take the complex conjugate of equation (21) and sandwich it between ϵ^{-1} and ϵ . The right-hand side becomes:

$$\Lambda^\mu_\nu v^\nu \epsilon^{-1} \sigma_\mu^* \epsilon = \Lambda^\mu_\nu v^\nu \bar{\sigma}_\mu$$

where we used one of the properties given in the text. The left-hand side becomes:

$$v^\mu \epsilon^{-1} \Lambda_L^* \sigma_\mu^* \Lambda_L^T \epsilon = v^\mu \epsilon^{-1} \Lambda_L^* \epsilon^{-1} \sigma_\mu^* \epsilon^{-1} \Lambda_L^T \epsilon = v^\mu \Lambda_R \bar{\sigma}_\mu \Lambda_R^\dagger$$

because $\epsilon^{-1} \Lambda_L^* \epsilon = \Lambda_R$, which also implies $\epsilon^{-1} \Lambda_L^T \epsilon = \Lambda_R^\dagger$. Therefore, we proved the equation outlined in the text.

- The last point derives from algebraic manipulations, given the last two point proven in the exercise.

The importance of this exercise lies in the fact that now we can construct spinor bilinears involving Weyl fermions and Dirac fermions, which can be used to write down interaction terms in the Lagrangian, for instance:

$$A^\mu \psi_L^\dagger \bar{\sigma}_\mu \psi_L, \quad A^\mu \psi_R^\dagger \sigma_\mu \psi_R$$

This is the way the Weyl spinors are coupled to the electromagnetic field.

Exercise 3

- Recalling the definition of $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$, one can easily check:

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \begin{cases} \mu = 0, \nu = 0 & \sigma^0 \bar{\sigma}^0 + \sigma^0 \bar{\sigma}^0 = 2, \\ \mu = 0, \nu = i & \sigma^0 \bar{\sigma}^i + \sigma^i \bar{\sigma}^0 = -\sigma^i + \sigma^i = 0, \\ \mu = i, \nu = j & \sigma^i \bar{\sigma}^j + \sigma^j \bar{\sigma}^i = -\sigma^i \sigma^j - \sigma^j \sigma^i = -\{\sigma^i, \sigma^j\} = -2\delta^{ij}. \end{cases}$$

In compact notation:

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2\eta^{\mu\nu}$$

Therefore, given the definition of the Dirac matrices, we can deduce the anticommutation relation:

$$\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} + (\mu \leftrightarrow \nu) = \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu \end{pmatrix} + (\mu \leftrightarrow \nu) = 2g^{\mu\nu} \mathbb{1}.$$

- Let us consider the second term in the right hand side of the equation:

$$\{\gamma^\mu, \gamma^5\} = i\gamma^\mu \gamma^0 \gamma^1 \gamma^2 \gamma^3 + i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\mu.$$

and let us bring γ^μ on the left of all the other matrices γ^ν , $\nu = 0, 1, 2, 3$. When $\mu \neq \nu$, the relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ gives $\gamma^\nu \gamma^\mu = -\gamma^\mu \gamma^\nu$, while when $\mu = \nu$ obviously $\gamma^\nu \gamma^\mu = \gamma^\mu \gamma^\nu$. Therefore $\{\gamma^\mu, \gamma^5\} = 0$.

- The only non-trivial point of the question is to show that $(\gamma^5)^2 = 1$. This can be worked out in a way similar to the one used in the previous point.