Quantum Field Theory

Set 10: solutions

Exercise 1

Since ϕ is a real field we have $\phi(x,t) = \phi(x,t)^*$. Using this equality in the inverse Fourier transform

$$\phi(k,t) = \int d^3x e^{-ikx}\phi(x,t)$$

we find

$$\phi(k,t)^* = \int d^3x e^{ikx}\phi(x,t)^* = \int d^3x e^{-i(-k)x}\phi(x,t) = \phi(-k,t).$$

The same reasoning can be used for $\pi(k,t)$. Now we plug the Fourier transform of $\phi(x,t)$ in the formula for the Hamiltonian. Consider the first term

$$\int d^3x \pi(x)^2 = \int d^3x \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3} e^{ix(k+q)}\pi(k)\pi(q).$$

We can now integrate in x to obtain a delta function that allows us to also integrate in q

$$\int d^3x \pi(x)^2 = \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}(2\pi)^3\delta(k+q)\pi(k)\pi(q) = \int \frac{d^3k}{(2\pi)^3}\pi(k)\pi(-k) = \int \frac{d^3k}{(2\pi)^3}|\pi(k)|^2,$$

where in the last equality we have used the identity just proved in the previous point. The same calculation can be used for the other two terms. The only small difference comes from the term with the gradient for which we have

$$\int d^3x(\nabla\phi(x))^2 = \int d^3x \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3} e^{ix(k+q)}(-(k\cdot q))\phi(k)\phi(q) = \int \frac{d^3k}{(2\pi)^3}k^2|\phi(k)|^2.$$

In the end we find

$$H = \int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}|\pi(k,t)|^2 + \frac{1}{2}(k^2+m^2)|\phi(k,t)|^2\right].$$

Exercise 2

The first commutator is

$$\begin{aligned}
[H, a(\vec{p})] &= \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left[a^\dagger(\vec{k})a(\vec{k}), a(\vec{p})\right] = \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left(a^\dagger(\vec{k})\left[a(\vec{k}), a(\vec{p})\right] + \left[a^\dagger(\vec{k}), a(\vec{p})\right]a(\vec{k})\right) \\
&= \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left(0 - (2\pi)^3\delta^3(\vec{p}-\vec{k})a(\vec{k})\right) = -\omega(\vec{p})a(\vec{p}).
\end{aligned}$$

Analogously,

$$\begin{aligned}
\left[H, a^\dagger(\vec{p})\right] &= \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left[a^\dagger(\vec{k})a(\vec{k}), a^\dagger(\vec{p})\right] = \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left(a^\dagger(\vec{k})\left[a(\vec{k}), a^\dagger(\vec{p})\right] + \left[a^\dagger(\vec{k}), a^\dagger(\vec{p})\right]a(\vec{k})\right) \\
&= \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\left(a^\dagger(\vec{k})(2\pi)^3\delta^3(\vec{p}-\vec{k}) + 0\right) = +\omega(\vec{p})a^\dagger(\vec{p}).
\end{aligned}$$

Exercise 3

We want to show the invariance under Lorentz transformations of the measure over momentum space $\frac{d^3k}{(2\pi)^3 2k_0}$, where $k_0 \equiv \omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2}$ is the energy associated to a given particle of mass m. There are two ways to achieve the result: the first consists in checking explicitly the invariance performing a Lorentz transformation on momenta; however we first prove it performing a manipulation. The measure can be rewritten as

$$\frac{d^3k}{(2\pi)^3 2k_0} = \frac{d^3k}{(2\pi)^3}\, dk_0\, \delta(k^2 - m^2)\theta(k_0) = \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k_0). \tag{1}$$

In order to convince oneself that this is true, one can consider a test function f of momenta and integrate it over k_0:

$$\int \frac{d^3k}{(2\pi)^3} dk_0 \delta(k_0^2 - |\vec{k}|^2 - m^2)\theta(k_0) f(\vec{k}, k_0) = \int \frac{d^3k}{(2\pi)^3} dk_0 \left(\frac{\delta(k_0 + \sqrt{|\vec{k}|^2 + m^2})}{2|k_0|} + \frac{\delta(k_0 - \sqrt{|\vec{k}|^2 + m^2})}{2|k_0|} \right) \theta(k_0) f(\vec{k}, k_0),$$

where we have used the well known relation for the δ function: given a function $g(x)$ which vanishes in the points $\{x_1, ..., x_n\}$, then

$$\delta(g(x)) = \sum_{i=1}^{n} \delta(x - x_i) \frac{1}{|g'(x_i)|}.$$

In the present case the equation $(k_0)^2 - |\vec{k}|^2 - m^2 = 0$ admits two opposite solutions, hence the two terms in the parenthesis, but the theta function gets rid of the second one since in that case $k_0 < 0$. Finally, integrating on k_0 one gets the initial measure:

$$\int \frac{d^3k}{(2\pi)^3} dk_0 \frac{\delta(k_0 - \sqrt{|\vec{k}|^2 + m^2})}{2|k_0|} \theta(k_0) f(\vec{k}, k_0) = \int \frac{d^3k}{(2\pi)^3 2\sqrt{|\vec{k}|^2 + m^2}} f(\vec{k}, \sqrt{|\vec{k}|^2 + m^2}).$$

Notice that the integrated function depends only on $\vec{k}$ (and the measure we are considering is defined on three-momenta). We have formally extended it to be a function of $\vec{k}$, k_0 but this is only a trick because the δ function forces the variables to be related by the *mass shell condition* $k^2 = m^2$.
The form of the measure we got allows us to show immediately the invariance under a Lorentz transformations.

- d^4k is invariant since the jacobian determinant of the change of variables is 1.
- $\delta(k^2 - m^2)$ is a function of the scalar $k^2 = k_\mu k^\mu$ and therefore it's itself invariant.
- The theta function is not a priori invariant under Lorentz transformations: it is so only if the sign of k_0' is the same of that of k_0. Lorentz transformations in general don't preserve the sign of the 0-component of a four vector: for example if $v_\mu = (1, 0, 0, 2)$, then one can easily find a boost in the third direction that makes $v_0' < 0$:
$$v_0' = \gamma v_0 - \beta\gamma v_3 = (1 - 2\beta)\gamma \implies \beta \geq \frac{1}{2}.$$
However one has to recall that the mass-shell condition $k_0 = \sqrt{|\vec{k}|^2 + m^2}$ makes k_μ a timelike fourvector (a fourvector in which the 0-component is larger than the modulus of its spatial threevector), that is to say a vector which lies inside the future lightcone centered in the origin. Transformations of the orthochronous Lorentz group, defined by the condition ${\Lambda^0}_0 > 0$ (those non containing the time reversal) send the future lightcone in itself and therefore if the four vector k_μ have positive 0-component, the same will be for k_μ'.
One can prove this explicitly. The transformed 0-component of k^μ is $k'^0 = {\Lambda^0}_0 k^0 + \sum_i {\Lambda^i}_0 k^i$, with $k^0 \geq \sqrt{\sum_i (k^i)^2}$, due to the mass-shell condition. The defining relation $\eta_{\mu\nu}{\Lambda^\mu}_\rho{\Lambda^\nu}_\sigma = \eta_{\rho\sigma}$ implies ${\Lambda^0}_0 > \sqrt{\sum_i ({\Lambda^i}_0)^2}$. Moreover, since $\sum_i {\Lambda^i}_0 k^i \geq -\sqrt{\sum_i (k^i)^2}\sqrt{\sum_i ({\Lambda^i}_0)^2}$, then:
$$(k^0)' \geq {\Lambda^0}_0 k^0 - \sqrt{\sum_i (k^i)^2}\sqrt{\sum_i ({\Lambda^i}_0)^2} \geq \left({\Lambda^0}_0 - \sqrt{\sum_i ({\Lambda^i}_0)^2} \right) k^0 > 0$$
implying that the sign of k'^0 is the same as the one of k^0.
Therefore the theta function will be left invariant in the distribution (1).

One can also check it explicitly performing Lorentz transformations: clearly the measure $\frac{d^3k}{(2\pi)^3 2k_0}$ is invariant under space rotations since d^3k is so and k_0 is a scalar under $SO(3)$. Therefore only pure boosts are left to check. Consider then a boost in the direction $\vec{n}$ with rapidity η (recall that the rapidity is defined as $\eta = \tanh^{-1}(\beta)$): we decompose the spatial momentum $\vec{k}$ in its longitudinal (i.e. along $\vec{n}$) and transverse (i.e. orthogonal to $\vec{n}$) parts: $\vec{k} = \vec{k}_T + \vec{k}_L$. Then the transformed quantities are:

$$\begin{aligned}
k_0' &= k_0 \cosh(\eta) + |\vec{k}_L| \sinh(\eta), \\
\vec{k}_L' &= k_0 \vec{n} \sinh(\eta) + \vec{k}_L \cosh(\eta), \\
\vec{k}_T' &= \vec{k}_T.
\end{aligned}$$

Note that the direction of $\vec{k}_L$ is fixed to be parallel to $\vec{n}$, therefore one can remove the symbol of vector and consider only the modulus k_L. One can as well decompose the differential $d^3k \longrightarrow d^2k_T\, dk_L$; therefore the measure transforms as:

$$\begin{aligned}
d^2k_T' &= d^2k_T, \\
dk_L' &= \frac{\partial k_L'}{\partial k_L} dk_L = \frac{\partial}{\partial k_L}(k_0 \sinh(\eta) + k_L \cosh(\eta)) dk_L = \frac{\partial}{\partial k_L}(\sqrt{k_L^2 + |\vec{k}_T|^2 + m^2}\ \sinh(\eta) + k_L \cosh(\eta)) dk_L \\
&= \left(\frac{k_L}{\sqrt{k_L^2 + |\vec{k}_T|^2 + m^2}} \sinh(\eta) + \cosh(\eta) \right) dk_L = \frac{1}{k_0} \left(k_L \sinh\eta + k_0 \cosh\eta \right) dk_L = \frac{k_0'}{k_0} dk_L.
\end{aligned}$$

At the end the ratio $\frac{d^3k}{(2\pi)^3 2k_0}$ is invariant also under Lorentz boosts.

Finally one can consider the distribution $d^3k\delta^3(\vec{k})$. The fastest way to see that it is invariant is to see it as the result of an integration over k_0:

$$d^3k\delta^3(\vec{k}) = \int dk^0 d^3k\, \delta^4(k) = \int d^4k\, \delta^4(k) \tag{2}$$

The expression on the right-hand side is Lorentz invariant because $d^4k' = |J(\Lambda)| d^4k$ and $\delta^4(k') = \delta^4(k)|J(\Lambda)|^{-1}$, where $J(\Lambda)$ is the determinant of the Jacobian of the Lorentz transformation.
Finally, the distribution $(2\pi)^3 k^0 \delta^3(\vec{k})$ is Lorentz-invariant because it's obtained by dividing the two invariant distributions $d^3k\delta^3(\vec{k})$ and $\frac{d^3k}{(2\pi)^3 k^0}$.

Exercise 4

Let's consider the two lines $\Sigma_1 = \{(t,x) : t = 0\}$ and $\Sigma_2 = \{(t,x) : t' = 0 \implies t = \beta x\}$. We can integrate the equation for the conservation of the current on the 2D section of the plane V between Σ_1 and Σ_2 as shown in Fig. 1.

We can write this integral in two different ways depending on whether we integrate first on t or on x:

$$\iint_V = \int_{-\infty}^{\infty} dt \int_{t/\beta}^{\infty} dx = \int_{-\infty}^{\infty} dx \int_0^{\beta x} dt$$

We will use both these forms when computing our result to eliminate the derivatives of the components of the current. We have

$$\begin{aligned}
0 = \iint_V \left[\partial_t J^0(t,x) + \partial_x J^1(t,x) \right] &= \int_{-\infty}^{\infty} dt \int_{t/\beta}^{\infty} dx\, \partial_x J^1(t,x) + \int_{-\infty}^{\infty} dx \int_0^{\beta x} dt\, \partial_t J^0(t,x) \\
&= -\int_{-\infty}^{\infty} dt\, J^1(t, t/\beta) + \int_{-\infty}^{\infty} dx\, \left[J^0(\beta x, x) - J^0(0,x) \right],
\end{aligned}$$

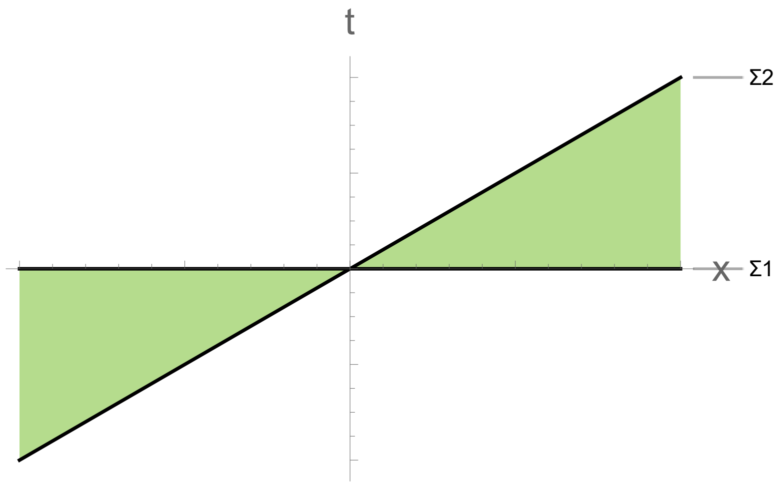

Figure 1:

where we have used the fact that the current vanishes for $x \to \infty$. The integral of the last term gives us the charge Q. We now want to prove that the sum of the other two is equal to Q'. First we change the integration variable of the first integral to x using $t = \beta x$.

$$0 = \int_{-\infty}^{\infty} dx \, \left[J^0(\beta x, x) - \beta J^1(\beta x, x)\right] - Q.$$

Then we change the variable again to $x' = \gamma(x - \beta t) = \frac{x}{\gamma}$ (we used $t = \beta x$) to find

$$0 = \int_{-\infty}^{\infty} dx \, \frac{\gamma}{\gamma} \left[J^0(t, x) - \beta J^1(t, x)\right] - Q = \int_{-\infty}^{\infty} dx' \, J'^0(t' = 0, x') - Q = Q' - Q \implies Q = Q'.$$

Exercise 5

Consider the following complex scalar field doublet:

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \qquad \phi_i \in \mathbb{R}.$$

Let us write the most general Lagrangian with terms of dimension $d \leq 4$ which is invariant under $\Phi \to U\Phi$, where $U \in SU(2)$. The result is the Lagrangian given in the text:

$$\mathcal{L} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2. \tag{3}$$

Invariance follows from $\Phi^\dagger \Phi \to \Phi^\dagger (U^\dagger U)\Phi = \Phi^\dagger \Phi$ since $U^\dagger U = \mathbb{1}$.
To see whether eq. (3) corresponds to a *reasonable* theory, let us compute the Hamiltonian density:

$$\mathcal{H} = \partial_0 \Phi^\dagger \partial_0 \Phi + \partial_i \Phi^\dagger \partial_i \Phi + m^2(\Phi^\dagger \Phi) + \lambda(\Phi^\dagger \Phi)^2. \tag{4}$$

For a theory to be well defined, we need the Hamiltonian to be bounded from below. Indeed when couplings to ordinary 'healthy' matter are taken into account the system is unstable: with zero net energy one can excite both sectors, the positive energy one and the negative energy one, without bound. In a quantum system this translates into an instability of the vacuum. The decay probability of the vacuum to a state involving negative energy excitations would be infinite, since the phase space is, and consequently no Lorentz invariant stable vacuum could exist. Looking at the Hamiltonian (4), the kinetic term is always positive and we can focus to constant field configurations. The potential is instead unbounded for $\lambda < 0$ when $(\Phi^\dagger \Phi) = const. \to \infty$, hence we need:

$$\lambda \geq 0.$$

Even if we obtained (3) by demanding $SU(2)$ invariance (and restricting to $d \leq 4$ terms), the resulting Lagrangian is invariant under a bigger symmetry group. First it is simple to check that $U(1)$ phase transformations $\Phi \to e^{i\alpha}\Phi$

are a symmetry, hence (3) is invariant *at least* under $U(1)\times SU(2)=U(2)$. In fact the symmetry group is even bigger. To see this let us write $\Phi^\dagger\Phi$ in terms of ϕ_i:

$$\Phi^\dagger\Phi=\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^4=\sum_{i=1}^4\phi_i^2.$$

Similarly we have

$$\partial_\mu\Phi^\dagger\partial^\mu\Phi=\sum_{i=1}^4\partial_\mu\phi_i\partial^\mu\phi_i.$$

We then can conclude that (3) is invariant under the group $O(4)$, under which $\phi_i\to\sum_{j=1}^4 O_{ij}\phi_j$, $O\in O(4)$. Indeed

$$\sum_{i=1}^4\phi_i^2\to\sum_{i=1}^4\sum_{j=1}^4\sum_{k=1}^4(O_{ij}\phi_j)(O_{ik}\phi_k)=\sum_{j=1}^4\sum_{k=1}^4\phi_j\underbrace{(O^TO)_{jk}}_{=\delta_{jk}}\phi_k=\sum_{j=1}^4\phi_j^2.$$

Invariance of the kinetic term follows in the same way.

Let us now add higher dimensional terms to (3) which respect the $U(2)$ symmetry. First it is easy to check that there are no dimension five terms which are both Lorentz and $U(2)$ invariant. At dimension 6 we have only one possibility with no derivatives

$$\mathcal{O}_1=(\Phi^\dagger\Phi)^3.$$

At dimension six we can also write terms with two derivatives and four fields. Then we find

$$\begin{aligned}
&\mathcal{O}_2=(\Phi^\dagger\Phi)(\partial_\mu\Phi^\dagger\partial^\mu\Phi),\\
&\mathcal{O}_3=(\Phi^\dagger\partial_\mu\Phi)(\Phi^\dagger\partial^\mu\Phi)+c.c.=(\Phi^\dagger\partial_\mu\Phi)(\Phi^\dagger\partial^\mu\Phi)+(\partial_\mu\Phi^\dagger\Phi)(\partial^\mu\Phi^\dagger\Phi),\\
&\mathcal{O}_4=i\left[(\Phi^\dagger\partial_\mu\Phi)(\Phi^\dagger\partial^\mu\Phi)-c.c.\right]=i(\Phi^\dagger\partial_\mu\Phi)(\Phi^\dagger\partial^\mu\Phi)-i(\partial_\mu\Phi^\dagger\Phi)(\partial^\mu\Phi^\dagger\Phi),\\
&\mathcal{O}_5=(\partial_\mu\Phi^\dagger\Phi)(\Phi^\dagger\partial^\mu\Phi).
\end{aligned}$$

Notice that $(\Phi^\dagger\partial_\mu\Phi)^*=(\partial_\mu\Phi^\dagger\Phi)$. Then these are found just taking all possible combinations of four fields where derivatives act on different fields and requiring reality. Terms where two derivatives act on the same field can be obtained from these adding a total derivative, hence we do not need to include them. For instance:

$$(\Phi^\dagger\Phi)(\Phi^\dagger\partial^2\Phi)+(\Phi^\dagger\Phi)(\partial^2\Phi^\dagger\Phi)=-2\mathcal{O}_2-\mathcal{O}_3-2\mathcal{O}_5+\partial_\mu\left[(\Phi^\dagger\Phi)(\Phi^\dagger\partial^\mu\Phi)+c.c.\right].$$

We neglect terms with four derivatives.
Now we can add to (3) these terms. Since $[\mathcal{L}]=4$, the coupling in front the $d=6$ terms must have dimension of an inverse mass square. Hence the modification of (3) induced by the addiction of these can always be written as

$$\mathcal{L}\to\mathcal{L}+\delta\mathcal{L}=\mathcal{L}+\frac{1}{M^2}\sum_{i=1}^5\lambda_i\mathcal{O}_i, \tag{5}$$

where the λ_i are dimensionless and M is a mass.
By construction (5) is still $U(2)$ invariant. Without doing further computations we can also easily see that $\mathcal{O}_1$ and $\mathcal{O}_2$ are $O(4)$ invariant, since they are built with the same building blocks present in (3). However it is possible to see that $\mathcal{O}_3$, $\mathcal{O}_4$, and $\mathcal{O}_5$ are not $O(4)$ invariant. Consider indeed

$$\begin{aligned}
(\Phi^\dagger\partial_\mu\Phi)&=\phi_1\partial_\mu\phi_1+i\phi_1\partial_\mu\phi_2-i\phi_2\partial_\mu\phi_1+\phi_2\partial_\mu\phi_2+\phi_3\partial_\mu\phi_3+i\phi_3\partial_\mu\phi_4-i\phi_4\partial_\mu\phi_3+\phi_4\partial_\mu\phi_4\\
&=\sum_{i=1}^4\phi_i\partial_\mu\phi_i+i\sum_{i,j=1}^2\epsilon_{ij}\phi_i\partial_\mu\phi_j+i\sum_{i,j=3}^4\epsilon_{ij}\phi_i\partial_\mu\phi_j.
\end{aligned}$$

Plugging this into the explicit expression of $\mathcal{O}_5$, for instance, we find

$$\mathcal{O}_5=\left(\sum_{i=1}^4\phi_i\partial_\mu\phi_i\right)^2+\left(\sum_{i,j=1}^2\epsilon_{ij}\phi_i\partial_\mu\phi_j+\sum_{i,j=3}^4\epsilon_{ij}\phi_i\partial_\mu\phi_j\right)^2.$$

The second term in the r.h.s. is not $O(4)$ invariant. We conclude that generically the modified Lagrangian (5) is $U(2)$ but not $O(4)$ invariant.