Quantum Field Theory

Set 12

Exercise 1: The Pauli–Lubanski (pseudo)vector

The Poincare group has two casimirs, P^2 and W^2, where P^μ and W^μ are respectively the momentum and the Pauli-Lubanski vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \,.$$

- Show that this definition is equivalent to

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma} \,.$$

- Calculate and express the result of the following commutators in terms of the Pauli-Lubanski vector
  1. $P_\mu W^\mu$,
  2. $[P^\mu, W^\nu]$,
  3. $[J^{\mu\nu}, W^\rho]$ (Hint: Use the identity $\eta^{\mu\nu}\epsilon^{\rho\alpha\beta\sigma} = \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\sigma} + \eta^{\mu\alpha}\epsilon^{\rho\nu\beta\sigma} + \eta^{\mu\beta}\epsilon^{\rho\alpha\nu\sigma} + \eta^{\mu\sigma}\epsilon^{\rho\alpha\beta\nu}$),
  4. $[W^\mu, W^\nu]$.
- Show that W^2 is a Casimir of the Poincare group.

Exercise 2: Vector spinor bilinears

Consider a Lorentz transformation:

$$\Lambda^\mu{}_\nu = e^{\omega^\mu{}_\nu}, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

where the boost and rotation parameter η^k, θ^k are defined as:

$$\omega^{ij} = \epsilon^{ijk}\theta^k, \quad \omega^{i0} = \eta^i$$

- Write explicitly the infinitesimal transformation for the components v^0, v^i of a Lorentz vector v^μ in terms of θ^k, η^k
- The $(1/2, 0)$, $(0, 1/2)$ representations of the Lorentz group are:

$$\Lambda_L(\vec{\eta}, \vec{\theta}) = e^{-\frac{1}{2}(\vec{\eta}+i\vec{\theta})\cdot\vec{\sigma}}$$

$$\Lambda_R(\vec{\eta}, \vec{\theta}) = e^{-\frac{1}{2}(-\vec{\eta}+i\vec{\theta})\cdot\vec{\sigma}}$$

  Consider the following action on a generic 2×2 hermitian matrix:

$$v'^\mu\sigma_\mu = \Lambda_L(\vec{\eta}, \vec{\theta}) v^\mu \sigma_\mu \Lambda_L^\dagger(\vec{\eta}, \vec{\theta})$$

  Prove at the infinitesimal level that this defines a Lorentz transformation on v^μ with parameters η^k, θ^k, where $\sigma_\mu \equiv (\mathbb{1}, -\vec{\sigma})$.

- Starting from the previous result show that also:

$$v'^{\mu}\bar{\sigma}_{\mu} = \Lambda_R(\vec{\eta},\vec{\theta})v^{\mu}\bar{\sigma}_{\mu}\Lambda_R^{\dagger}(\vec{\eta},\vec{\theta})$$

defines the same Lorentz transformation on v^{μ}, where $\bar{\sigma}_{\mu} \equiv (\mathbb{1}, \vec{\sigma})$. You may make use of the following properties of the $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ matrix:

$$\epsilon^T = \epsilon^{-1}, \quad \epsilon^{-1}\sigma^i\epsilon = -(\sigma^i)^*, \quad \Lambda_R = \epsilon^{-1}\Lambda_L^*\epsilon$$

- Repeat for:

$$v'^{\mu}\gamma_{\mu} = \Lambda_D v^{\mu}\gamma_{\mu}\Lambda_D^{-1}.$$

where Λ_D are Lorentz representation matrices acting on Dirac spinors:

$$\Lambda_D = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}$$

and the Dirac γ matrices are:

$$\gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \bar{\sigma}_{\mu} & 0 \end{pmatrix}$$

Note: Pay attention to the position of the Lorentz indices throughout the whole exercise.

Exercise 3: Clifford algebra

- Compute the anticommutator of two Dirac matrices: $\{\gamma^{\mu}, \gamma^{\nu}\}$
- Define the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. Using the previous result convince yourself that $\{\gamma^{\mu}, \gamma^5\} = 0$
- Prove that $P_L \equiv \frac{1}{2}(1+\gamma^5)$, $P_R \equiv \frac{1}{2}(1-\gamma^5)$ define two orthogonal projectors: $P_L + P_R = 1$, $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = 0$.