Quantum Field Theory

Set 10

Exercise 1: Hamiltonian in momentum space

Consider the Fourier expansions at a given time for a real scalar field $\phi(x,t)$ and its momentum $\pi(x,t)$:

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3} e^{ikx}\phi(k,t), \qquad \pi(x,t) = \int \frac{d^3k}{(2\pi)^3} e^{ikx}\pi(k,t)$$

and the Hamiltonian

$$H = \int d^3x \left[\frac{1}{2}\pi(x,t)^2 + \frac{1}{2}(\nabla\phi(x,t))^2 + \frac{1}{2}m^2\phi(x,t)^2\right]$$

- show that $\phi(k,t) = \phi^*(-k,t)$ and $\pi(k,t) = \pi^*(-k,t)$,
- compute H in terms of the Fourier transformed field and momentum.

Exercise 2: Ladder operators

Given the algebra of the ladder operators (non relativistic normalization)

$$\left[a(\vec{q}), a^\dagger(\vec{p})\right] = (2\pi)^3\delta^3(\vec{q}-\vec{p}),$$

and the expression of the Hamiltonian

$$H = \int \frac{d^3k}{(2\pi)^3}\,\omega(\vec{k})\,a^\dagger(\vec{k})a(\vec{k}),$$

compute the commutation relations

$$[H, a(\vec{p})] = ? \qquad \left[H, a^\dagger(\vec{p})\right] = ?$$

Exercise 3: Invariant measure

Consider the measure on thee dimensional momentum space $\frac{d^3k}{(2\pi)^3 2k^0}$, where $k_0 = \sqrt{|\vec{k}|^2 + m^2}$.

- Performing a boost in a particular direction, say $\hat{k}_3$, show the invariance of the measure under Lorentz transformations.
  (*Hint*: use the form of the boost transformations $k_0' = \cosh(\eta)\ k_0 + \sinh(\eta)\ k_3\ ,\ k_3' = \cosh(\eta)\ k_3 + \sinh(\eta)\ k_0$.
- Show that the product $\delta^3(\vec{k})d^3k$ is invariant under Lorentz transformation and deduce that the distribution $(2\pi)^3 2k^0\delta^3(\vec{k})$ is invariant as well. (*Hint*: write the product $\delta^3(\vec{k})d^3k$ in an explicitly covariant way involving all the four coordinates).

Exercise 4: Lorentz invariance of Noether charges

Consider a generic field theory in $1+1$ dimensions. A Noether current $J^\mu(t,x)$ is associated to each symmetry. Then, given a coordinate system $\{t,x\}$, we define the Noether charge integrating the zero component of the current in space at fixed time $t=0$:

$$Q = \int dx\, J^0(0,x).$$

Since the Noether current satisfies $\partial_\mu J^\mu = \partial_t J^0 + \partial_x J^1 = 0$, Q is independent on time. This definition however is not manifestly Lorentz invariant. Indeed consider a different coordinate system $\{t', x'\}$ related to the previous one through a boost transformation:

$$t' = \gamma(t - \beta x), \qquad\qquad x' = \gamma(x - \beta t).$$

In these coordinates, we would naturally define the Noether charge integrating in space at fixed $t' = \gamma(t - \beta x) = 0$:

$$Q' = \int dx' \, J'^0(0, x'),$$

where $J'^\mu(t', x') = \Lambda^\mu_\nu J^\nu(t, x)$ is the Noether current in the second reference frame.

- Show that $Q' = Q$ (*Hint*: integrate the identity in $\partial_\mu J^\mu = 0$ in the plane section enclosed by the lines $t = 0$ and $t' = 0$).

Exercise 5: Custodial symmetry and vacuum stability

Consider the following Lagrangian:

$$\mathcal{L} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2, \tag{1}$$

where Φ is a complex scalar field doublet:

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \qquad \phi_i \in \mathbb{R}.$$

- What constraint must λ satisfy in order to obtain a *reasonable* theory?
- Find the global symmetries of (2). (*Hint*: write $\Phi^\dagger\Phi$ in terms of the ϕ_i).
- Add to (2) all possible dimension 6 terms invariant under $\Phi \to U\Phi$, $U \in U(2)$ which contain at most two derivatives. Does the resulting Lagrangian have the same symmetries of (2)?