Quantum Field Theory

Homework Set

This is a more advanced exercise set than the one solved in the exercise sessions, which addresses important topics of the course. Solving it is a good preparation for the exam and for the QFT2 class.

Exercise 1: Isospin $\times$ Lorentz spinors

Consider a Weyl spinor transforming in the representation $(1/2, 0)$ of the Lorentz group ψ_L:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu},$$
$$\psi'_L(x') = \Lambda_L \psi_L(\Lambda^{-1} x'),$$

where Λ_L is the Lorentz transformation in the representation $(1/2, 0)$. We have omitted the spinor index α for shortness.
Identify the representation of the Lorentz group the following term belongs to:

$$\psi_L^{\dagger} \bar{\sigma}^{\mu} \psi_L.$$

Consider now a pair of *left* Weyl spinors: ψ_L^1, ψ_L^2 and assume they form a dublet Ψ_L of an additional *Isospin* $SU(2)$ symmetry (don't confuse this $SU(2)$ with the Lorentz Group, it's an internal symmetry). In addition take a single *right* Weyl spinor ψ_R which is a singlet of the $SU(2)$ Isospin (ψ_R, Ψ_L are completely unrelated fields, they are not obtained one from the other using ε). Note that the doublet $\Psi_L = \binom{\psi_L^1}{\psi_L^2}$ has two indices

$$(\Psi_L)^a_{\alpha} \qquad \begin{array}{c} \alpha = 1,2 \text{ represents the Lorentz index} \\ a = 1,2 \text{ represents the Isospin index} \end{array}$$

while ψ_R has only the Lorentz $\dot{\beta}$ index, and each transformation acts separately on the two indices:

$$\text{Lorentz} \begin{cases} x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \\ (\Psi'_L)^a_{\alpha}(x') = (\Lambda_L)_{\alpha}{}^{\beta} (\Psi_L)^a_{\beta}(\Lambda^{-1} x') \\ (\psi'_R)^{\dot{\beta}}(x') = (\Lambda_R)^{\dot{\beta}}{}_{\dot{\delta}} (\psi_R)^{\dot{\delta}}(\Lambda^{-1} x'), \end{cases}$$

$$\text{Isospin} \begin{cases} x'^{\mu} = x^{\mu} \\ (\Psi'_L)^a_{\alpha}(x') = U^a_b (\Psi_L)^b_{\alpha}(x) \\ (\psi'_R)^{\dot{\beta}}(x') = (\psi_R)^{\dot{\beta}}(x). \end{cases}$$

- Identify the representation of Isospin and Lorentz group the following terms belong to:

  $$\psi_R^{\dagger} \Psi_L\,, \qquad [(\Psi_L)^a]^{\dagger} (\sigma^i)^a{}_b \not{\partial} \Psi_L^b,$$

  where we have suppressed the Lorentz indices and here $\not{\partial} \equiv \bar{\sigma}_{\mu} \partial^{\mu}$. Note that in the second term the Pauli matrix is contracted with the Isospin indices while $\not{\partial}$ with the Lorentz ones.

Answer: $\psi_R^{\dagger} \Psi_L \sim \frac{1}{2} \otimes (0,0)$; $[(\Psi_L)^a]^{\dagger} (\sigma^i)^a{}_b \not{\partial} \Psi_L^b \sim 1 \otimes (0,0)$.

Exercise 2: Decomposition of tensor products

SU(2)

If u_a and v_b are doublets of SU(2), decompose into irreducible representations the following tensor products.

- $u_a v_b^* \sim \frac{1}{2} \otimes \overline{\frac{1}{2}}$;
- $u_a v_b \sim \frac{1}{2} \otimes \frac{1}{2}$.

Lorentz

If ψ_α and ϕ_β are left handed spinors and A^μ and B^ν are 4-vectors, decompose into irreducible representations the following tensor products:

- $\psi_\alpha \phi_\beta \sim (\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)$;
- $A_\mu \psi_\alpha \sim (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, 0)$;
- $A_\mu B_\nu \sim (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})$. Focus only on the scalar irrep. Can you guess the others?

Exercise 3: Scale transformations

The action of a free real massless scalar field in d dimension reads:

$$\mathcal{S} = \int dt d^{d-1}x \ \mathcal{L}(t, x), \qquad \mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi,$$

where $\mu = 0, 1, .., d-1$, $x^0 = t$ and the indices are raised and lowered with the d-dimensional metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, .., -1)$.
Consider the transformation

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = e^\lambda x^\mu \\ \phi(x) &\longrightarrow \phi'(x') = e^{k\lambda} \phi(e^{-\lambda} x') \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{R}$.

- Compute the value of k such that the above transformation defines a symmetry of the theory.
- For $d = 4$ compute the energy momentum tensor $T^\mu{}_\nu$ of the theory. Compute the trace of the energy momentum tensor.
- Consider the *improved energy momentum tensor* $K^\mu{}_\nu = T^\mu{}_\nu + A \delta^\mu_\nu \Box \phi^2 + B \partial^\mu \partial_\nu \phi^2$. Show, in $d = 4$, that we can choose the values of A and B in such a way that $\partial_\mu K^\mu{}_\nu = 0$ and $K^\mu{}_\mu = 0$.
- For $d = 4$ compute the Noether's current S^μ associated to the symmetry defined in (1). Express it in terms of the improved energy momentum tensor $K^\mu{}_\nu$ and show which constraints $\partial_\mu S^\mu = 0$ imposes on $K^\mu{}_\nu$.
- Find for which values of d the addition of the following potentials to the free Lagrangian density doesn't spoil the symmetry (that is to say the transformation is still a symmetry of the new Lagrangian density):

  $$\mathcal{L}' = \partial^\mu \phi \partial_\mu \phi - \begin{cases} \frac{m^2}{2!} \phi^2, \\ \text{or} \\ \frac{\beta}{3!} \phi^3, \\ \text{or} \\ \frac{\alpha}{4!} \phi^4. \end{cases}$$

  and compute the dimension (in powers of energy) of the parameters m, β, α for those values of d.
- Discuss this result.

Exercise 4: Charges Algebra

Consider a Lie symmetry group $\mathcal{G}$ described by parameters $\{\alpha^i\}$, acting on coordinates and fields as

$$\begin{aligned} g: &\qquad x^\mu \longrightarrow x'^\mu = x^\mu, \\ g: &\qquad \phi_a(x) \longrightarrow \phi'_a(x') = \mathcal{R}(g)_a{}^b \phi_b(x), \end{aligned}$$

where $\mathcal{R}(g)_a{}^b$ is the representation of an element $g \in \mathcal{G}$.
Show that the Noether's charges Q_i together with the product defined by the Poisson brackets form an Algebra which is isomorphic to the Lie Algebra of $\mathcal{G}$.