Quantum Field Theory

Set 2

Exercise 1: Causality

Show that the propagator of a free relativistic point particle ($E_p = \sqrt{\vec{p}^2 + m^2}$)

$$K(\vec{x}, t; \vec{0}, 0) \equiv \langle \vec{x} | e^{-iHt} | \vec{0} \rangle \;, \tag{1}$$

does not vanishing outside the lightcone ($|\vec{x}| > t$). Focus on the limit $|\vec{x}| \gg t$ and estimate the integral through the method of the stationary phase.

Exercise 2

Consider the process $\gamma + e^- \to \gamma + e^- + e^+ + e^-$. In the frame where the electron is at rest (laboratory frame) compute the minimum photon energy E_γ necessary for this process to happen.

Exercise 3

Consider an infinite system of particles located along the direction x at distance a from one another. Each particle has mass m and is connected to the neighbor masses by a spring of elastic constant k and vanishing rest length. The masses are allowed to move only in the direction y. Call y_i the displacement of the mass at $x = i \cdot a$.
1. Find the Lagrangian that describes the system and the relative equations of motion.
2. Find the Hamiltonian and the first order equations of motion. Verify that they give rise to the same second order equations of motion.
3. Perform the continuum limit in the Lagrangian ($a \to 0$) keeping the following quantities constant:

- $\mu = \frac{m}{a}$ = mass per unit length of the chain of oscillator,
- $Y = k\, a$ = Young elastic coefficient.

4. Perform the same limit in the equations of motion and verify that they can be obtained as Euler-Lagrange equations of the Action.
5. Find the general solution of the equations of motion.
6. What happens if in the initial discrete model each particle is also connected to the point ($x = ia$, $y = 0$) by a spring of frequency w'? How does the solution change in the continuum limit? (keep w'=constant in the limit)

Exercise 4: Higher derivative scalar theory

Consider the *Lagrangian density* of a real scalar field $\phi(\vec{x}, t)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

- Find the equation of motion of the field ϕ
- Specialize to the massive $\lambda\phi^4$-theory: $V = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$.

Add to the lagrangian density the term: $\alpha(\partial_\mu\phi\partial^\mu\phi)^2$

- Compute the dimension of the constant α.
- Find the equation of motion of the field ϕ.

Exercise 5: Hamiltonian formalism

Consider the action of the *massive* $\lambda\phi^4$ *scalar theory*:

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right] .$$

- Find the equation of motion using the Hamiltonian formalism and show the equivalence with the Lagrangian formalism.