Quantum Field Theory

Set 9

Exercise 1: WZW term

Consider 5 real scalar fields, η_i ($i = 1, 2$) and ϕ_I ($I = 1, 2, 3$), with Lagrangian

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{2}\partial_\mu \eta_i \partial^\mu \eta_i + \frac{1}{2}\partial_\mu \phi_I \partial^\mu \phi_I - \frac{m^2}{2}(\eta_i \eta_i) - \frac{M^2}{2}(\phi_I \phi_I) \\
& - \lambda(\eta_i \partial_\mu \eta_j)(\partial_\nu \phi_I \partial_\rho \phi_J \partial_\sigma \phi_K)\epsilon^{ij}\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma}
\end{aligned}
$$

where ϵ^{ij}, ϵ^{IJK}, $\epsilon^{\mu\nu\rho\sigma}$ are the Levi-Civita antisymmetric tensors in respectively 2, 3 and 4 dimensions.

- What is the dimensionality of the coupling λ?
- Find the symmetries of the system

  Hint: consider a generic transformation $\eta_i \to M_{ij}\eta_j$ and $\phi_i \to N_{ij}\phi_j$ where M and N are 2×2 and 3×3 matrices respectively. What properties should M and N satisfy to leave the Lagrangian invariant? Also remember that for a $N \times N$ matrix M we have the following identity

  $$\epsilon_{i_1, i_2, \ldots, i_N} M_{i_1 j_1} M_{i_2 j_2} \ldots M_{i_N j_N} = \det(M)\epsilon_{j_1, j_2, \ldots, j_N}$$

- On which parameters do the Noether currents depend on?
- Suppose now that the η fields are complex, and the Lagrangian is:

  $$
  \begin{aligned}
  \mathcal{L} =& \frac{1}{2}\partial_\mu \eta_i^* \partial^\mu \eta_i + \frac{1}{2}\partial_\mu \phi_I \partial^\mu \phi_I - \frac{m^2}{2}(\eta_i^* \eta_i) - \frac{M^2}{2}(\phi_I \phi_I) \\
  & - i\lambda(\eta_i^* \partial_\mu \eta_i)(\partial_\nu \phi_I \partial_\rho \phi_J \partial_\sigma \phi_K)\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma}
  \end{aligned}
  $$

  What are in this case the symmetries of the system?
- What if we change the interaction term to:

  $$\lambda(\eta_i \partial_\mu \eta_j)(\partial_\nu \phi_I \partial_\rho \phi_J \partial_\sigma \phi_K)\epsilon^{ij}\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma} + h.c.$$

Exercise 2: $SO(N)$ and $\mathbb{D}_4$

Part 1. Consider N scalar fields $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_n(x))$ transforming in the fundamental representation of $SO(N)$:

$$\Phi(x) \longrightarrow \Phi'(x) = O\Phi(x), \qquad O \in SO(N).$$

Write the most general Lagrangian $\mathcal{L}$ for $\Phi(x)$ such that

- $\mathcal{L}$ is Lorentz invariant,
- contains only terms with dimension less or equal than four (or equivalently, whose coupling have dimension greater or equal to zero),
- it is invariant under $SO(N)$.

Is the internal symmetry group of $\mathcal{L}$ bigger than $SO(N)$? Consider the case $N = 1$ and $N \geq 2$ separately.

Part 2. Consider now a theory with two scalar fields ϕ_1 and ϕ_2. Build the most general Lorentz invariant Lagrangian with terms up to dimension 4 that is symmetric under the following three transformations separately:

- $\phi_1 \to -\phi_1$
- $\phi_2 \to -\phi_2$
- $\phi_1 \leftrightarrow \phi_2$

What is the difference between this group and $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$? (Reminder: $\mathbb{Z}_2$ is the group formed by two elements $\{1, -1\}$ with their product)
How many elements does this transformation group have?
Can you find a matrix representation for this group?
In which case the Lagrangian you just built is invariant under $O(2)$ as in the first part of this exercise?

Exercise 3: Decomposition in Fourier modes

Given the decomposition of a real scalar field in a finite cubic volume $V = L^3$

$$\phi(t, \vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^3} \frac{1}{\sqrt{V}} \, \phi_n(t) \, e^{i \frac{2\pi}{L} \vec{n} \cdot \vec{x}} \tag{1}$$

Expand $\int d^3x (\vec{\nabla}\phi)^2(t, \vec{x})$ in Fourier modes $\phi_n(t)$.

Exercise 4: Noether's current: a different approach

There is another way to find the Noether's current of a Lagrangian with a given symmetry without having to use the explicit formula of the theorem. This can sometimes be faster than applying the formula. The idea is that we consider a transformation of the symmetry group where we take the Lie parameters α dependent on x (local transformation). We know that if α does not depend on x (global transformation) then $\delta S = 0$ under this transformation by definition of symmetry. If α depends on x then δS will no longer be zero, but at the leading order in α it will be proportional to $\partial\alpha$. Explicitly

- Compute δS as in the derivation of the Noether theorem but being careful that α depends on x. Knowing that δS is zero when α is constant show that

$$\delta S = \int d^4x \, (\partial_\mu \alpha(x)) J^\mu + \mathcal{O}(\alpha^2)$$

  where J^μ is the Noether current given by the Noether theorem (you can work in the case $K = 0$ for simplicity).

- Apply this procedure to the $U(1)$ symmetry in a complex scalar field theory and show that you get the same current as the previous exercise set.

Exercise 5: Noether's charge as generator of transformation: part 2

Given a Lagrangian density $\mathcal{L}(\phi_a, \partial_\mu \phi_b)$ consider the global symmetry transformation defined by

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = x^\mu \\ \phi_a(x) &\longrightarrow \phi'_a(x) = D[\phi]_a(x) \simeq \phi_a(x) + i\alpha^i T^b_{i,a} \phi_b(x), \end{aligned}$$

where the $T^b_{i,a}$ are the representation of the generators of some Lie algebra of some Lie group $\mathcal{G}$.

- Find the following transformation of the conjugate momenta $\pi^a(x)$

$$\pi^a(x) \longrightarrow \pi'^a(x) = D^\pi[\pi]^a(x) \simeq \pi^a(x) - i\alpha^i T^a_{i,b}\pi^b(x),$$

  and show that charge Q_i built starting from the Noether's current is the generator of the transformation:

$$\delta_\alpha \pi^a(x) \equiv \pi'^a(x) - \pi^a(x) = \alpha^i \{Q_i, \pi^a(x)\} = -i\alpha^i T^a_{i,b}\pi^b(x)\,.$$

- Using the Jacobi identity and that $\{Q_i, \phi_a\} = iT^b_{i,a}\phi_b(x)$ (as you proved in the last exercise set), compute the following object

$$\{\phi_a \{Q_i, Q_j\}\}\ , \tag{2}$$

  and deduce $\{Q_i, Q_j\}$.

- Compute explicitly the Poisson bracket

$$\{Q_i, Q_j\} \equiv \int d^3z \frac{\delta Q_i}{\delta \pi_a(z)} \frac{\delta Q_j}{\delta \phi^a(z)} - \frac{\delta Q_i}{\delta \phi_a(z)} \frac{\delta Q_j}{\delta \pi^a(z)} \tag{3}$$

  and check that the result is consistent with what you have found before.