# Quantum Field Theory

## Set 7

### Exercise 1: $(0, 1/2)$ and $(1/2, 0)$ representations of the Lorentz group.

Starting from the explicit form of the spin-1/2 representations of $SU(2)$ determine the form of the Lorentz transformations in the $(0, 1/2)$ and $(1/2, 0)$ representations. (Hint: How are the generators of the Lorentz group represented in these representations?). What about the $(1/2, 1/2)$ representation?

### Exercise 2: Representation of Poincaré Group on scalar functions (class repeat)

We define a *scalar* $f$ as a map from events $e$ in Minkowski space to real numbers:

$$f : e \in \mathbb{M} \to f(e) \in \mathbb{R}$$

In a specific frame, we label an event $e$ with a coordinate four-vector $x^\mu \in \mathbb{R}^{3,1}$. The realization of $f$ in this frame is given by a scalar function $\phi$ of $x^\mu$: $f(e) = \phi(x(e)) \equiv \phi(x)$. The realization of $f$ in a different frame $x'^\mu$ is generically given by a different function $\phi'$ given by $f(e) = \phi'(x'(e)) \equiv \phi'(x')$.
Call $H$ the set of scalar functions of $x^\mu$ and denote $\phi(x)$ a generic element of it. Take an element of the Poincaré group acting on $x^\mu$ as:

$$g \equiv (\Lambda, a) : \; x^\mu \longrightarrow \Lambda^\mu{}_\nu \, x^\nu + a^\mu \; \equiv P_{(\Lambda,a)}(x^\mu).$$

Define the action of $g$ on $H$ as follows

$$\begin{aligned} &(\mathcal{D}_{(\Lambda,0)}[\phi])(x) = \phi(\Lambda^{-1}x)\,, \qquad (\mathcal{D}_{(0,a)}[\phi])(x) = \phi(x-a)\,, \\ &\phi'(x) \equiv (\mathcal{D}_{(\Lambda,a)}[\phi])(x) = (\mathcal{D}_{(0,a)}\left[\mathcal{D}_{(\Lambda,0)}[\phi]\right])(x) = \phi(\Lambda^{-1}(x-a)). \end{aligned}$$

Show that this definition respects the group composition law and therefore defines a representation.

### Exercise 3: Poincaré algebra on fields

Consider a scalar field $\phi(x)$. As shown in class, the Poincaré generators are represented on scalar fields by

$$P^\mu = i\partial^\mu, \qquad J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu).$$

Compute the commutators

$$[P^\mu, P^\nu] = ?, \qquad [P^\mu, J^{\rho\sigma}] = ?, \qquad [J^{\mu\nu}, J^{\rho\sigma}] = ?,$$

using this representation. Show that you get the Poincaré algebra.

### Exercise 4: Lorentz invariance of Lagrangians

Consider a real scalar field $\phi(x)$. Write how the field $\phi(x)$ transforms under the action of an element of the Poincaré group. Take the action $S$ built starting from the following Lagrangian density

$$S = \int dt\, d^3x \; \mathcal{L}\,, \qquad \mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - V(\phi).$$

Show that:

- the volume element $dtd^3x$ is invariant under Poincaré transformations;

- the Poincaré group is a symmetry of the theory.

Repeat the analysis for the electromagnetic field:

- identify how the vector field $A_\mu$ transforms under the action of an element of the Poincaré group;
- show that the Poincaré group is a symmetry of the theory $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$.