Quantum Field Theory

Set 4

Exercise 1: Permutation group $\mathcal{S}_3$

Consider the set of all permutations of the configuration (a, b, c). This is known as the permutation group $\mathcal{S}_3$.

- What is the order $|\mathcal{S}_3|$ of the group?
- Represent the objects $\{1, 2, 3\}$ as vectors in $\mathbb{R}^3$,

$$|1\rangle = (1, 0, 0), \qquad |2\rangle = (0, 1, 0), \qquad |3\rangle = (0, 0, 1).$$

  Write the corresponding matrix representation of $\mathcal{S}_3$. You should write $|\mathcal{S}_3|$ matrices.
- Show that the representation you constructed is *reducible*. In particular, can you guess non-trivial $\{\alpha_1, \alpha_2, \alpha_3\}$ such that $\alpha_1 |1\rangle + \alpha_2 |2\rangle + \alpha_3 |3\rangle$ remains invariant under any permutation? The vector space spanned by this vector is an *invariant subspace*.
- Find the orthogonal vector space. Show that is also an invariant subspace.
- Compute the *equivalent* representation to the one you constructed in the basis that spans each invariant subspace. Verify that it is block diagonal, where each block is irreducible. This shows that the original representation is *completely reducible*.

Exercise 2: Some Lie Groups

Consider the following Lie groups given here by their defining representations:

$$\begin{aligned}
U(N) &= \left\{ N \times N \text{ complex matrices such that } U^\dagger U = UU^\dagger = 1 \right\}, \\
SU(N) &= \left\{ N \times N \text{ complex matrices such that } U^\dagger U = UU^\dagger = 1, \, \det U = 1 \right\}, \\
O(N) &= \left\{ N \times N \text{ real matrices such that } R^T R = R^T R = 1 \right\}, \\
SO(N) &= \left\{ N \times N \text{ real matrices such that } R^T R = R^T R = 1, \, \det R = 1 \right\}, \\
SL(N, \mathbb{C}) &= \{ N \times N \text{ complex matrices such that } \det A = 1 \ \}.
\end{aligned}$$

Consider transformations close to the identity and expand a general element of the group as a power series in the Lie parameters

$$D[g(\alpha)] = \mathbb{1} + i\alpha^i T^i + O(\alpha^2).$$

For each of the given groups find the constraints on the T^i and deduce the dimension of the group.

Hint: you will have to use the following property:

$$\det(\mathbb{1} + i\alpha^i T^i + O(\alpha^2)) = 1 + i\alpha^i \mathrm{Tr}(T^i) + O(\alpha^2)$$

Exercise 3: Schur's lemma

Suppose that there are two irreducible inequivalent representations D_1 and D_2. Let A be a linear operator which commutes with the direct sum of these two representations $D_1 \oplus D_2$. Show that $A = \lambda_1 \mathbb{1} \oplus \lambda_2 \mathbb{1}$.

Exercise 4: Jacobi Identity

Show that, for any Lie Algebra defined by the commutation relations

$$\left[T^a, T^b\right] = i f^{abc} T^c \ ,$$

the following *Jacobi identity* holds:

$$\sum_d \left(f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd}\right) = 0.$$

This is a straightforward consequence of the identity:

$$\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] = 0.$$