Quantum Field Theory

Set 3

Exercise 1: Hamiltonian formalism

Consider the action of the *massive* $\lambda\phi^4$ *scalar theory*:

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right] .$$

- Find the equation of motion using the Hamiltonian formalism and show the equivalence with the Lagrangian formalism.

Exercise 2: Classical Electromagnetism

Consider the classical electromagnetic fields $\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$.

- Write the Maxwell equations in presence of external an source.

Define the *field strength* $F_{\mu\nu}$ as the 4×4 matrix

$$F_{\mu\nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{bmatrix} \qquad F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix}$$

- Recalling the definitions of the field in term of the vector potential $A_\mu = (A_0,\, A_i)$ show that
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
- Introduce the *Lagrangian density* of the electromagnetic field in the presence of an external source:
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$
where J^μ is the current associated to the source. .
Find the Euler Lagrange equations and show that they correspond to the inhomogeneous Maxwell equations.
- Show that the additional two Maxwell equations follow from the *Bianchi identity*
$$\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0$$
- Find the static solution (independent of time) $A_\mu(\vec{x})$ for the current corresponding to a particle at rest $J^\mu = (e\delta^3(\vec{x}-\vec{x}_0), \vec{0})$.

Exercise 3: Affine transformations

Consider the two parameter Lie group acting on the real numbers defined by the transformation

$$U(\alpha,\beta):\ x \to x' = e^\alpha x + \beta$$

for any $(\alpha,\beta) \in \mathbb{R}^2$.

Show that this transformation defines a group. Is it an abelian group?

Exercise 4: Dilation symmetry

Consider a particle with position $q(t)$ and mass m obeying the differential equation

$$m\ddot{q}(t) + kq^2(t) = 0$$

with k a constant. What is the dimension of k?

Find the value of the constant p such that the equation is symmetric under the dilation transformation

$$q(t) \to q'(t') = \lambda^{-p} q(t), \quad t' = \lambda t$$

Could you have guessed the result from the dimension of k?

Exercise 5 [optional]

The Large Hadron Collider (LHC) started its operations on 10th September 2008, completing the first entire revolution of the 27 km long ring at 10:28 a.m. The proton beam coming from the SPS has been injected inside the LHC with an energy of 450 GeV per proton. Compute the velocity of the beam in units of c and the Lorentz γ factor.
At full performance at LHC, proton beams circulate in opposite directions with an energy of 7 TeV each. There are approximately 2.8×10^{14} protons per beam. Find the velocity in km/h of a running TGV with mass 400 tons, so that its kinetic energy is the same of that of a proton beam in LHC.
Suppose that the LHC, instead of being a collider with two proton beams circulating in opposite directions with an energy of 7 TeV each, were a collider with fixed target and an incoming proton beam with an energy E_a in the laboratory frame. What should E_a be in order to have the same energy of LHC in the center of mass frame?
Given that the power lost by a charged particle in a circular trajectory of radius r is given by

$$P = \frac{2e^2}{3r^2}(\beta\gamma)^4$$

and that we can give approximately 1 GeV of energy per particle every lap, what is the minimum radius of this accelerator to reach the center of mass energy of 1.5 TeV?