Exercises in preparation for the exam

Exercise 1: Consider a field theory with a vector field A_μ and 3 scalars ϕ_I $(I = 1, 2, 3)$ with lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu \phi_I \partial^\mu \phi_I - \frac{M^2}{2}(\phi_I \phi_I) \\ &- \lambda A_\mu (\partial_\nu \phi_I \partial_\rho \phi_J \partial_\sigma \phi_K)\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma}\end{aligned}$$

where ϵ^{IJK}, $\epsilon^{\mu\nu\rho\sigma}$ are the Levi-Civita antisymmetric tensors in respectively 3 and 4 dimensions.

- what is the dimensionality of the coupling λ
- discuss the symmetries of the system

Exercise 2: Consider a relativistic field theory with $U(1)$ symmetry and the following field content: one left-handed Weyl spinors ψ_L, one right-handed Weyl spinor ψ_R and a complex scalar ϕ. The corresponding $U(1)$ charges of the fields are $(q_{\psi_L}, q_{\psi_R}, q_\phi) = (1, -1, 2)$.

- Write the most general Lagrangian (kinetic terms plus interactions) invariant under $U(1)$
- Suppose now, instead, to have the two mentioned spinors plus another left-handed spinor χ_L (not the scalar). The charges are respectively $(q_{\psi_L}, q_{\chi_L}, q_{\psi_R}) = (1, -1, -1)$. Write the most general mass terms for the spinors, while preserving the $U(1)$ symmetry.

Hint: Recall that $\Lambda_L^\dagger = \Lambda_R^{-1}$ and $\Lambda_L^{-1} = \epsilon^{-1}\Lambda_L^T\epsilon$

Exercise 3: Consider a real neutral scalar field. The ladder operators satisfy:

$$[a(\mathbf{k}), a^\dagger(\mathbf{p})] = (2\pi)^3 2E_k \delta^3(\mathbf{k} - \mathbf{p})\,.$$

Given the following state,

$$|\psi\rangle = \int d\phi d\theta \sin\theta \left(e^{2i\phi}\sin^2\theta\, a^\dagger(\mathbf{k}_{\theta,\phi})|0\rangle\right),$$

where $\mathbf{k}_{\theta,\phi} = |\mathbf{k}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$,

- show that

$$J^3|\psi\rangle = 2|\psi\rangle \qquad \text{and} \qquad J^i J^i|\psi\rangle = 6|\psi\rangle\,,$$

where $J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$ is the angular momentum operator:

$$J^{ij} = -i\int d\Omega_k \left[a^\dagger(\mathbf{k})\left(k^i\frac{\partial}{\partial k^j} - k^j\frac{\partial}{\partial k^i}\right)a(\mathbf{k})\right].$$

Hint: Notice that $e^{i\phi}\sin\theta = (k^1 + ik^2)/|\mathbf{k}|$.