
RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 14

Fall 2022

1. The Hulse-Taylor Binary

- (a) Approximating the orbit as circular, and the two stars' masses as equal, we get that the orbit is simply parametrized as

$$\begin{aligned}x_1 &= r \cos(\Omega t), & y_1 &= r \sin(\Omega t), \\x_2 &= -r \cos(\Omega t), & y_2 &= -r \sin(\Omega t).\end{aligned}\tag{1}$$

- (b) The energy density is thus

$$\begin{aligned}T^{00}(t, \vec{x}) &= M\delta(z) \left[\delta(x - r \cos(\Omega t))\delta(y - r \sin(\Omega t)) \right. \\&\quad \left. + \delta(x + r \cos(\Omega t))\delta(y + r \sin(\Omega t)) \right].\end{aligned}\tag{2}$$

- (c) The quadrupole moments are thus

$$\begin{aligned}I_{xx} &= 2Mr^2 \cos^2(\Omega t), \\I_{xy} &= 2Mr^2 \cos(\Omega t) \sin(\Omega t), \\I_{yy} &= 2Mr^2 \sin^2(\Omega t), \\I_{xz} &= 0, \\I_{yz} &= 0, \\I_{zz} &= 0,\end{aligned}\tag{3}$$

and $I_{ji} = I_{ij}$.

- (d) To compute the power emitted we first compute the traceless part of the quadrupole moment

$$\begin{aligned}J_{xx} &= I_{xx} - \frac{1}{3}(I_{xx} + I_{yy}) = \frac{2Mr^2}{3} [3 \cos^2(\Omega t) - 1], \\J_{xy} &= I_{xy} = 2Mr^2 \cos(\Omega t) \sin(\Omega t), \\J_{yy} &= I_{yy} - \frac{1}{3}(I_{xx} + I_{yy}) = \frac{2Mr^2}{3} [3 \sin^2(\Omega t) - 1], \\J_{zz} &= -\frac{1}{3}(I_{xx} + I_{yy}) = -\frac{2Mr^2}{3}.\end{aligned}\tag{4}$$

where all other entries are related by symmetry or are vanishing. The third derivatives are

$$\frac{d^3 J_{ij}}{dt^3} = 8M\Omega^3 r^2 \begin{pmatrix} \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}\tag{5}$$

The object of interest to compute the power is

$$\begin{aligned}\frac{d^3 J_{ij}}{dt^3} \frac{d^3 J^{ij}}{dt^3} &= \left(\frac{d^3 J_{xx}}{dt^3} \right)^2 + \left(\frac{d^3 J_{yy}}{dt^3} \right)^2 + 2 \left(\frac{d^3 J_{xy}}{dt^3} \right)^2 \\ &= 64M^2\Omega^6 r^4 \left(2\sin^2(2\Omega t) + 2\cos^2(2\Omega t) \right) \\ &= 128M^2\Omega^6 r^4\end{aligned}\tag{6}$$

The emitted power is thus

$$P = -\frac{128}{5}GM^2r^4\Omega^6.\tag{7}$$

- (e) We derived the rate of change of the total energy of the system. To find the rate of change of the period, we need to find the relation between period and energy of the system. For a circular orbit, we can equate the Newtonian force to the centripetal force to find the velocity in terms of the distance between the stars

$$\frac{GM^2}{(2r)^2} = \frac{Mv^2}{r}\tag{8}$$

which gives us the velocity

$$v = \sqrt{\frac{GM}{4r}}.\tag{9}$$

From this we can derive the total energy of the system as a function of the distance between the two stars only

$$E = Mv^2 - \frac{GM^2}{2r} = -\frac{GM^2}{4r}.\tag{10}$$

We can invert this to find the radius of an orbit of energy E

$$r = -\frac{GM^2}{4E}.\tag{11}$$

The period is instead

$$T = \frac{2\pi r}{v} = 2\pi r \sqrt{\frac{4r}{GM}} = \frac{\pi GM}{2} \left(-\frac{M}{E} \right)^{\frac{3}{2}}.\tag{12}$$

Its rate of change is thus

$$\frac{dT}{dt} = \frac{3\pi G}{4} \left(-\frac{M}{E} \right)^{\frac{5}{2}} \frac{dE}{dt} = \frac{3G}{(2\pi^2)^{1/3}} \left(\frac{T}{GM} \right)^{\frac{5}{3}} P.\tag{13}$$

We need to write the power itself in terms of the period

$$P = -\frac{128}{5}GM^2r^4\Omega^6 = -\frac{128}{5}2^{\frac{2}{3}}G^{\frac{7}{3}}\left(\frac{\pi M}{T}\right)^{\frac{10}{3}}.\tag{14}$$

In total, the differential equation for T is

$$\frac{dT}{dt} = -\frac{384}{5}2^{\frac{1}{3}}\pi^{\frac{8}{3}}\left(\frac{GM}{T}\right)^{\frac{5}{3}}.\tag{15}$$

Let us restore the powers of c . The quantity in brackets should clearly be dimensionless, which is achieved by adding c^3 in the denominator

$$\frac{dT}{dt} = -\frac{384}{5} 2^{\frac{1}{3}} \pi^{\frac{8}{3}} \left(\frac{GM}{Tc^3} \right)^{\frac{5}{3}}. \quad (16)$$

The solution to this differential equation is

$$T(t) = \left(T_0^{\frac{8}{3}} - \frac{1024}{5} 2^{\frac{1}{3}} \pi^{\frac{8}{3}} \left(\frac{GM}{c^3} \right)^{\frac{5}{3}} t \right)^{\frac{3}{8}}. \quad (17)$$

By plugging in the numbers, we find that the lifetime of this system, or t_* such that $T(t_*) = 0$, is

$$t_* \sim 1 \text{ Gyrs}, \quad (18)$$

which is only off by a factor of ~ 3 from the real world system (which has an elliptical orbit) and has a lifetime of $t_* \sim 300 \text{ Myrs}$.

2. The Inside of a Fermi Star

- (a) For convenience, let us write the TOV equation in terms of the function $m(r)$ only. We use the following relations

$$\rho(r) = \frac{1}{4\pi r^2} \frac{dm}{dr}, \quad P = \frac{\rho}{3}. \quad (19)$$

The TOV equation reads

$$\frac{2m'(12m + (2m' - 3)r)}{r(r - 2m)} + 3m'' = 0, \quad (20)$$

where primes are derivatives with respect to r .

- (b) By substituting $m(r) = Ar$ we get

$$\begin{aligned} \frac{2A(12Ar + (2A - 3)r)}{r(r - 2Ar)} &= 0 \\ rA(28A - 6) &= 0 \\ A &= \frac{3}{14}. \end{aligned} \quad (21)$$

The solution is thus $m(r) = \frac{3}{14}r$. The energy density and pressure will then be

$$\rho(r) = \frac{3}{56\pi r^2}, \quad P(r) = \frac{1}{56\pi r^2}. \quad (22)$$

The central density and pressure are infinite, and the radius is infinite since $P(r)$ is zero only at infinity.

(c) To write the metric we need to compute $\alpha(r)$.

$$\alpha(r) = \int dr \frac{m + 4\pi r^3 P}{r(r - 2m)} = \int dr \frac{1}{2r} = \frac{1}{2} \ln(r) + c. \quad (23)$$

Gluing it to the external Schwarzschild metric we get

$$\begin{aligned} \frac{1}{2} \ln(R) + c &= \frac{1}{2} \ln \left(1 - \frac{2M}{R} \right) \\ c &= \frac{1}{2} \ln \left(\frac{1}{R} - \frac{2M}{R^2} \right), \end{aligned} \quad (24)$$

where we are keeping R around as a regulator, even though it is technically infinity. The $\alpha(r)$ function is thus

$$\alpha(r) = \frac{1}{2} \ln \left(\frac{r}{R} - \frac{2Mr}{R^2} \right). \quad (25)$$

The metric inside the star is thus

$$\begin{aligned} ds^2 &= -e^{2\alpha} dt^2 + \left(1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 d\Omega_2^2 \\ &= - \left(\frac{r}{R} - \frac{2Mr}{R^2} \right) dt^2 + \frac{7}{4} dr^2 + r^2 d\Omega_2^2. \end{aligned} \quad (26)$$

To simplify our lives we redefine the coordinates as $\bar{r} = \frac{\sqrt{7}}{2} r$. The spatial part of the metric becomes

$$ds_3^2 = d\bar{r}^2 + \frac{7}{4} \bar{r}^2 d\Omega_2^2. \quad (27)$$

Let us then compute the length of a circumference of fixed radius $\bar{r} = a$ around the center of the star. For simplicity, we look at the circumference at the equator, where $\theta = \frac{\pi}{2}$. The relevant line element is

$$ds = \sqrt{g_{\phi\phi}} d\phi = r d\phi = \frac{2}{\sqrt{7}} \bar{r} d\phi \quad (28)$$

We obtain

$$C = \frac{2}{\sqrt{7}} \int_0^{2\pi} a d\phi = \frac{4}{\sqrt{7}} \pi a. \quad (29)$$

Clearly, the angle corresponding to a full circle around the origin is $\frac{4}{\sqrt{7}} \pi$ rather than 2π . This is a diagnostic of the presence of a conical singularity. Its presence is an artifact due to the infinite central density in this model.

(d) The radius of the star is defined as the place where $P(r) = 0$. Obviously, that happens only at $r = \infty$ in this model. The conceptual mistake is to say that the equation of state of such a system remains $P = \frac{\rho}{3}$ all the way to infinity, while instead at some point the fermions will have to be treated non-relativistically, leading to a different equation of state and a finite mass.