RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 7 Fall 2023

1. Hyperbolic Space

(a) Simply plug in the proposed coordinate choices in the hyperboloid constraint and check that it is verified.

(b) Start from the fact that the metric in embedding space is the 3 dimensional $\eta_{\mu\nu}$ and use the formula for how the metric components change under a coordinate transformation

$$g_{\mu'\nu'} = \frac{\partial X^{\mu}}{\partial x^{\mu'}} \frac{\partial X^{\nu}}{\partial x^{\nu'}} \eta_{\mu\nu} \,. \tag{1}$$

(c) To compute the length of such a segment, we need to compute

$$\Delta s = \int_{\text{segment}} ds \,. \tag{2}$$

The vector that points in the direction of the segment is ∂_z . We thus act with the metric tensor on two copies of that vector

$$g(\partial_z, \partial_z) = \frac{L^2}{z^2} \,. \tag{3}$$

The length of the segment is thus

$$\Delta s = \int_{z_1}^{z_2} \frac{L}{z} \mathrm{d}z = L \ln\left(\frac{z_2}{z_1}\right) \,. \tag{4}$$

A generic line that ends at $z_2 = 0$ has thus infinite length.

(d) Starting with geodesic equation for x^{ρ}:

$$\frac{\mathrm{d}^2 x^{\rho}}{d\lambda^2} = -\Gamma^{\rho}_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{d\lambda} \frac{\mathrm{d}x^{\nu}}{d\lambda} \tag{5}$$

one expands the right hand side to get

$$-\Gamma^{\rho}_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{d\lambda} \frac{\mathrm{d}x^{\nu}}{d\lambda} = -\Gamma^{\rho}_{\mu\nu} v^{\mu} v^{\nu} \left(\frac{\mathrm{d}x^{\bar{\sigma}}}{d\lambda}\right)^2 \tag{6}$$

The left hand side is bit more complex

$$\begin{aligned}\frac{\mathrm{d}^2 x^{\rho}}{d\lambda^2} = \frac{\mathrm{d}}{d\lambda}\left(\left(\frac{\mathrm{d}x^{\bar{\sigma}}}{d\lambda}\right) \frac{\mathrm{d}x^{\rho}}{\mathrm{d}x^{\bar{\sigma}}}\right) = \frac{\mathrm{d}^2 x^{\bar{\sigma}}}{d\lambda^2} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}x^{\bar{\sigma}}} + \frac{\mathrm{d}x^{\bar{\sigma}}}{d\lambda} \frac{\mathrm{d}}{d\lambda} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}x^{\bar{\sigma}}} = \\ = \frac{\mathrm{d}^2 x^{\bar{\sigma}}}{d\lambda^2} v^{\rho} + \left(\frac{\mathrm{d}x^{\bar{\sigma}}}{d\lambda}\right)^2 \frac{\mathrm{d}^2 x^{\rho}}{\mathrm{d}x^{\bar{\sigma}2}}\end{aligned} \tag{7}$$

Inserting the geodesic equation for $\frac{\mathrm{d}^2 x^{\bar{\sigma}}}{d\lambda^2}$ one gets

$$\frac{\mathrm{d}^2 x^{\bar{\sigma}}}{d\lambda^2} v^\rho + \left(\frac{\mathrm{d}x^{\bar{\sigma}}}{d\lambda}\right)^2 \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}x^{\bar{\sigma}2}} = -\Gamma^{x^{\bar{\sigma}}}_{\mu\nu} \frac{\mathrm{d}x^\mu}{d\lambda}\frac{\mathrm{d}x^\nu}{d\lambda} v^\rho + \left(\frac{\mathrm{d}x^{\bar{\sigma}}}{d\lambda}\right)^2 \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}x^{\bar{\sigma}2}} =$$

$$= -\Gamma^{\bar{\sigma}}_{\mu\nu} v^\mu v^\nu \left(\frac{\mathrm{d}x^{\bar{\sigma}}}{d\lambda}\right)^2 v^\rho + \left(\frac{\mathrm{d}x^{\bar{\sigma}}}{d\lambda}\right)^2 \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}x^{\bar{\sigma}2}} \tag{8}$$

Assembling the left and right hand side together one can clean up $\left(\frac{\mathrm{d}x^{\bar{\sigma}}}{d\lambda}\right)^2$ prefactoring both sides

$$-\Gamma^{\bar{\sigma}}_{\mu\nu} v^\mu v^\nu v^\rho + \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}x^{\bar{\sigma}2}} = -\Gamma^\rho_{\mu\nu} v^\mu v^\nu \tag{9}$$

or equivalently

$$\frac{\mathrm{d}v^\rho}{\mathrm{d}x^{\bar{\sigma}}} = -\Gamma^\rho_{\mu\nu} v^\mu v^\nu + v^\rho\, \Gamma^{\bar{\sigma}}_{\mu\nu} v^\mu v^\nu \tag{10}$$

The non-vanishing Christoffel symbols in Poincarè half-plane are

$$\Gamma^x_{xz} = -\frac{1}{z}, \qquad \Gamma^z_{xx} = \frac{1}{z}, \qquad \Gamma^z_{zz} = -\frac{1}{z}. \tag{11}$$

Let's use $z' = \frac{\mathrm{d}z}{\mathrm{d}x}$, $z'' = \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}$. The geodesic equation is

$$z'' = -\Gamma^z_{xx} - \Gamma^z_{zz}(z')^2 + z' 2\Gamma^x_{xz} z' \tag{12}$$

$$z'' = -\frac{1}{z} + \frac{1}{z}(z')^2 - \frac{2}{z}(z')^2 \tag{13}$$

$$zz'' + (z')^2 = -1 \tag{14}$$

$$(zz')' = -1 \tag{15}$$

$$zz' = x_0 - x \tag{16}$$

where x_0 is the constant of integration. Continuing

$$zz' = x_0 - x \tag{17}$$

$$\frac{1}{2}\left(z^2\right)' = x_0 - x \tag{18}$$

$$\frac{1}{2}z^2 = -\frac{1}{2}(x_0 - x)^2 + \frac{1}{2}l^2 \tag{19}$$

having l^2 as another constant of integration. Finally one can clean up the formula a bit

$$z^2 + (x_0 - x)^2 = l^2 \tag{20}$$

to recognize that geodesics are circles centered on the $z = 0$ line. Moreover, for $x_0 \to \infty$ and $l \to \infty$ with $l - x_0$ fixed we get straight lines that are perpendicular to $z = 0$. These are also geodesics.

Alternative solution Let's show that AdS2 geodesics are straight-lines and semi-circles centered on the boundary using a more elegant method. Using Poincaré coordinates, the geodesic equations follow from variation of the action

$$S = \int d\lambda \left(\frac{\dot{z}^2}{z^2} + \frac{\dot{x}^2}{z^2}\right). \tag{21}$$

Variation with respect to x and z yield the geodesic equations

$$\frac{d}{d\lambda}\frac{\dot{x}}{z^2} = 0, \qquad \ddot{z} + \frac{\dot{x}^2 - \dot{z}^2}{z} = 0. \tag{22}$$

We can find a first solution by choosing $x = const$, which fixes $z(\lambda) = c_1 e^{c_2\lambda}$ for some constants c_1 and c_2. The only important property of $z(\lambda)$ is the image, or what kind of curve will (x, z) span on the Poincaré half-plane. We thus do not care about the constants c_1 and c_2, as long as we fix $c_1 > 0$ since that's the correct domain of z. In particular, let us choose $c_1 = c_2 = 1$ without loss of generality.

$$\gamma(\lambda) = \begin{pmatrix} z(\lambda) \\ x(\lambda) \end{pmatrix} = \begin{pmatrix} e^{\lambda} \\ k \end{pmatrix}, \tag{23}$$

This is the parametrization of a vertical line of constant x. Let us move on to the second family of geodesics.

There is an important property of the AdS metric in Poincaré coordinates, which is that the **inversion** operation $I : x^\mu \to x^\mu / x^2$ is an isometry. Another important property is that isometries map geodesics to geodesics. Therefore, applying the inversion to the vertical line, we obtain

$$\gamma \to \begin{pmatrix} z' \\ x' \end{pmatrix} = \frac{1}{e^{2\lambda} + k^2}\begin{pmatrix} e^{\lambda} \\ k \end{pmatrix}. \tag{24}$$

Now the claim is that this parametrizes circles. To see this we compute

$$z'^2 + x'^2 = \frac{1}{e^{2\lambda} + k^2} = \frac{x'}{k} \Leftrightarrow z'^2 + \left(x' - \frac{1}{2k}\right)^2 = \frac{1}{4k^2}, \tag{25}$$

which is indeed the equation of a circle of radius $\frac{1}{2k}$ centered at $z' = 0$ (the boundary) and $x' = \frac{1}{2k}$. Since $z' > 0$ it is only a half circle.

(e) Let's the write Killing equation for a vector V:

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0 \tag{26}$$

Explicitly the components are

$$\partial_x V_x - \Gamma^z_{xx} V_z = 0 \tag{27}$$
$$\partial_x V_z + \partial_z V_x - 2\Gamma^z_{xx} V_z = 0 \tag{28}$$
$$\partial_z V_z - \Gamma^z_{zz} V_z = 0 \tag{29}$$

Starting from (29):

$$\partial_z V_z + \frac{V_z}{z} = 0 \tag{30}$$

one gets

$$V_z = \frac{\alpha(x)}{z} \tag{31}$$

where $\alpha(x)$ is to be determined. One shall use that solution to solve (27)

$$\partial_x V_x - \frac{\alpha(x)}{z^2} = 0 \tag{32}$$

which readily integrates to

$$V_x = \frac{A(x)}{z^2} + B(z) \tag{33}$$

with $A'(x) = \alpha(x)$. Putting these solutions to (28):

$$\begin{aligned}
\partial_x \left(\frac{\alpha(x)}{z}\right) + \partial_z \left(\frac{A(x)}{z^2} + B(z)\right) + \frac{2}{z}\left(\frac{A(x)}{z^2} + B(z)\right) = 0 \\
\frac{A''(x)}{z} + \left(-2\frac{A(x)}{z^3} + \dot{B}(z)\right) + \left(2\frac{A(x)}{z^3} + \frac{B(z)}{z}\right) = 0 \\
A''(x) = -(z\dot{B}(z) + 2B(z))
\end{aligned}$$

As the left side is a function of x only, and the right side is a function of z only, they must both be equal to a constant. The solution to LHS is

$$A(x) = ax^2 + bx + c \tag{34}$$

where a, b, c are the other integration constants. The right hand side is then

$$z\dot{B}(z) + 2B(z) = -2a \tag{35}$$

solved in general by $B(z) = -a - \frac{d}{z^2}$. Therefore, the general solution is

$$V_x = \frac{ax^2 + bx + c}{z^2} - a - \frac{d}{z^2} \tag{36}$$

$$V_z = \frac{2ax + b}{z} \tag{37}$$

Note that solution depends only on $c-d$, not on c and d separately. Without loss of generality one may set $d = 0$. To get vector components, one multiplies by inverse metric:

$$V^x = z^2 V_x = a\left(x^2 - z^2\right) + bx + c \tag{38}$$

$$V^z = z^2 V_y = 2axz + bz \tag{39}$$

or

$$V = \left(a\left(x^2 - z^2\right) + bx + c\right)\partial_x + (2axz + bz)\,\partial_z \tag{40}$$

This is linear combination of 3 vector fields

$$P = \partial_x\,, \tag{41}$$

$$D = x\partial_x + z\partial_z\,, \tag{42}$$

$$K = \left(x^2 - z^2\right)\partial_x + 2xz\partial_z\,, \tag{43}$$

the first two represent, respectively, translations in x and dilatations. These make sense if you look back at how geodesics on the half plane look like. The third one is a more obscure isometry of hyperbolic space called **special conformal transformations**. They roughly correspond to performing an inversion $x^\mu \to \frac{x^\mu}{x^2}$, then a translation and then another inversion.

Let us study the algebra of these Killing vectors through their commutators

$$[D,P] = -\,(\partial_x x)\partial_x = -P\,, \tag{44}$$

$$[D,K] = \left((x\partial_x + z\partial_z)\left(x^2 - z^2\right) \right)\partial_x + \tag{45}$$

$$+\left((x\partial_x + z\partial_z)2xz \right)\partial_z + \tag{46}$$

$$-\left(\left(x^2 - z^2\right)\partial_x x \right)\partial_x + \tag{47}$$

$$-\left(2xz\partial_z z \right)\partial_z = K\,, \tag{48}$$

$$[P,K] = \left(\partial_x \left(x^2 - z^2\right) \right)\partial_x + \partial_x(2xz)\partial_z = 2D\,. \tag{49}$$

2. Lie Derivatives

(a) This follows from the definition

$$(\mathcal{L}_V W)^\nu = V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu = -(W^\mu \partial_\mu V^\nu - V^\mu \partial_\mu W^\nu) = -(\mathcal{L}_W V)^\nu\,. \tag{50}$$

(b) Transforming the vector coordinates we get

$$\begin{aligned}(\mathcal{L}_V W)^\nu =& V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu \\ =& \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} \frac{\partial x^{\rho'}}{\partial x^\mu} \partial_{\rho'} \left(\frac{\partial x^\nu}{\partial x^{\nu'}} W^{\nu'} \right) - \frac{\partial x^\mu}{\partial x^{\mu'}} W^{\mu'} \frac{\partial x^{\rho'}}{\partial x^\mu} \partial_{\rho'} \left(\frac{\partial x^\nu}{\partial x^{\nu'}} V^{\nu'} \right) \\ =& \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} \frac{\partial x^{\rho'}}{\partial x^\mu} \left(\frac{\partial^2 x^\nu}{\partial x^{\nu'} \partial x^{\rho'}} W^{\nu'} + \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial W^{\nu'}}{\partial x^{\rho'}} \right) \\ & - \frac{\partial x^\mu}{\partial x^{\mu'}} W^{\mu'} \frac{\partial x^{\rho'}}{\partial x^\mu} \left(\frac{\partial^2 x^\nu}{\partial x^{\nu'} \partial x^{\rho'}} V^{\nu'} + \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial V^{\nu'}}{\partial x^{\rho'}} \right) \\ =& V^{\rho'} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial W^{\nu'}}{\partial x^{\rho'}} - W^{\rho'} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial V^{\nu'}}{\partial x^{\rho'}} \\ =& \frac{\partial x^\nu}{\partial x^{\nu'}} (\mathcal{L}_V W)^{\nu'}\,.\end{aligned} \tag{51}$$

(c) Consider the scalar $f \equiv W^\mu T_\mu$. Because of the Leibniz rule,

$$\begin{aligned}(\mathcal{L}_V f) = (\mathcal{L}_V W^\mu T_\mu) &= W^\mu (\mathcal{L}_V T)_\mu + (\mathcal{L}_V W)^\mu T_\mu \\ &= W^\mu (\mathcal{L}_V T)_\mu + V^\rho \partial_\rho W^\mu T_\mu - W^\rho \partial_\rho V^\mu T_\mu\,.\end{aligned} \tag{52}$$

At the same time, since f is a scalar,

$$(\mathcal{L}_V f) = V^\mu W^\nu \partial_\mu T_\nu + V^\nu T_\mu \partial_\nu W^\mu\,. \tag{53}$$

Comparing, we get

$$\begin{aligned} W^\mu(\mathcal{L}_V T)_\mu &= V^\mu W^\nu \partial_\mu T_\nu + V^\nu T_\mu \partial_\nu W^\mu - V^\rho \partial_\rho W^\mu T_\mu + W^\rho \partial_\rho V^\mu T_\mu \\ &= V^\mu W^\nu \partial_\mu T_\nu + W^\rho \partial_\rho V^\mu T_\mu \,, \end{aligned} \tag{54}$$

so

$$(\mathcal{L}_V T)_\mu = V^\nu \partial_\nu T_\mu + T_\nu \partial_\mu V^\nu \tag{55}$$

(d) Use Leibniz' rule

$$\begin{aligned} (\mathcal{L}_V A)_{\mu\nu} &= (\mathcal{L}_V S)_\mu T_\nu + (\mathcal{L}_V T)_\nu S_\mu \\ &= (V^\rho \partial_\rho S_\mu + S_\rho \partial_\mu V^\rho) T_\nu + (V^\rho \partial_\rho T_\nu + T_\rho \partial_\nu V^\rho) S_\mu \\ &= V^\rho \partial_\rho A_{\mu\nu} + A_{\rho\nu} \partial_\mu V^\rho + A_{\mu\rho} \partial_\nu V^\rho \end{aligned} \tag{56}$$

(e) For the Lie derivative of a vector, this follows from what we showed for the commutator of two vectors. For the Lie derivative of a dual vector, we get

$$\begin{aligned} V^\nu \nabla_\nu T_\mu + T_\nu \nabla_\mu V^\nu &= V^\nu \partial_\nu T_\mu - \Gamma^\rho_{\nu\mu} T_\rho V^\nu + T_\nu \partial_\mu V^\nu + T_\nu \Gamma^\nu_{\mu\rho} V^\rho \\ &= V^\nu \partial_\nu T_\mu + T_\nu \partial_\mu V^\nu \,, \end{aligned} \tag{57}$$

where we used the symmetry of the lower indices of the connection.

(f) The metric is a symmetric $(0,2)$ tensor, and so we can use the identity for $(\mathcal{L}_V A)_{\mu\nu}$ that we proved before, with covariant derivatives in place of the partial derivatives.

$$\begin{aligned} (\mathcal{L}_K g)_{\mu\nu} &= K^\rho \nabla_\rho g_{\mu\nu} + g_{\rho\nu} \nabla_\mu K^\rho + g_{\mu\rho} \nabla_\nu K^\rho \\ &= g_{\rho\nu} g^{\rho\alpha} \nabla_\mu K_\alpha + g_{\mu\rho} g^{\rho\alpha} \nabla_\nu K_\alpha \\ &= \delta^\alpha_\nu \nabla_\mu K_\alpha + \delta^\alpha_\mu \nabla_\nu K_\alpha \\ &= 2\nabla_{(\mu} K_{\nu)} \,. \end{aligned} \tag{58}$$

So we proved that

$$(\mathcal{L}_K g)_{\mu\nu} = 0 \tag{59}$$

is equivalent to

$$\nabla_{(\mu} K_{\nu)} = 0 \,. \tag{60}$$

3. Killing Vectors on the Sphere

(a) From the round metric, we immediately read off the first Killing vector, here expressed in the coordinate basis:

$$K_\Phi = \partial_\phi, \tag{61}$$

because the metric components do not depend explicitly on ϕ. Notice that we use the capital Greek letter Φ to give a name to the Killing vector, not to name its components. Its components are infact given by

$$K^\theta_\Phi = 0 \,, \qquad K^\phi_\Phi = 1 \,. \tag{62}$$

To find the other two Killing vectors, consider the embedding of the unit sphere in $\mathbb{R}^3$:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} . \tag{63}$$

We know that rotations around the three axis are isometries of the sphere. K_Φ generates the rotations on the (x, y) plane, which in Cartesian coordinates are

$$K_\Phi = -y\partial_x + x\partial_y \,. \tag{64}$$

Then we also expect to have

$$K_{\mathrm{xz}} = z\partial_x - x\partial_z \,, \qquad K_{\mathrm{zy}} = -z\partial_y + y\partial_z \,. \tag{65}$$

But (x, y, z) are coordinates in this embedding $\mathbb{R}^3$ space. To relate these results to the coordinates on S^2, we use the inverse transformations

$$\begin{cases} \theta = \arctan\left(\sqrt{\frac{x^2+y^2}{z^2}}\right) \,, \\ \phi = \arctan\left(\frac{y}{x}\right) \,. \end{cases} \tag{66}$$

Using the chain rule, we get

$$\begin{aligned} \partial_x &= \frac{\partial\theta}{\partial x}\partial_\theta + \frac{\partial\phi}{\partial x}\partial_\phi \\ &= \frac{x}{r^2}\sqrt{\frac{z^2}{x^2+y^2}}\partial_\theta - \frac{y}{x^2+y^2}\partial_\phi \\ &= \frac{1}{r}\cos\phi\cos\theta\partial_\theta - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\partial_\phi \,. \end{aligned} \tag{67}$$

Analogously,

$$\partial_z = -\frac{1}{r}\sin\theta\partial_\theta \,, \qquad \partial_y = \frac{1}{r}\sin\phi\cos\theta\partial_\theta + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\partial_\phi \,. \tag{68}$$

Putting everything together, we get

$$\begin{aligned} K_{\mathrm{xz}} &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi \,, \\ K_{\mathrm{zy}} &= -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi \,. \end{aligned} \tag{69}$$

We can also introduce r as an extra auxiliary variable and simply use the chain rule!! The sphere is a maximally symmetric space. In $n = 2$, we expect 3 Killing vectors, which are thus given by the set $\{K_\Phi, K_{\mathrm{xz}}, K_{\mathrm{zy}}\}$.

(b) Computing the Lie brackets of these vectors, we get

$$\begin{aligned} [K_\Phi, K_{\mathrm{xz}}] &= K_{\mathrm{zy}} \,, \\ [K_{\mathrm{xz}}, K_{\mathrm{zy}}] &= K_\Phi \,, \\ [K_{\mathrm{zy}}, K_\Phi] &= K_{\mathrm{xz}} \,. \end{aligned} \tag{70}$$

These are precisely the commutation relations that define the $\mathfrak{so}(3)$ algebra. In general, *the Killing vector fields of a manifold are the infinitesimal generators of that manifold's isometry group.*

4. Maxwell's Stress Tensor

Varying the action with respect to the metric, we get

$$\begin{aligned}
\delta_g S_M &= \int d^n x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \frac{\delta\sqrt{-g}}{\delta g^{\alpha\beta}} - \sqrt{-g}\frac{1}{4} F_{\rho\sigma} F_{\mu\nu} \frac{\delta(g^{\rho\mu} g^{\sigma\nu})}{\delta g^{\alpha\beta}} \right) \delta g^{\alpha\beta} \\
&= \int d^n x \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \frac{1}{2\sqrt{-g}} \frac{\delta g}{\delta g^{\alpha\beta}} - \sqrt{-g}\frac{1}{4} F_{\rho\sigma} F_{\mu\nu} \left[g^{\rho\mu} \frac{\delta g^{\sigma\nu}}{\delta g^{\alpha\beta}} + g^{\sigma\nu} \frac{\delta g^{\rho\mu}}{\delta g^{\alpha\beta}} \right] \right) \delta g^{\alpha\beta} \\
&= \int d^n x \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \frac{-g}{2\sqrt{-g}} g_{\alpha\beta} - \sqrt{-g}\frac{1}{4} \left(F^{\mu}{}_{\alpha} F_{\mu\beta} + F_{\alpha}{}^{\nu} F_{\beta\nu} \right) \right) \delta g^{\alpha\beta} \\
&= \frac{1}{4} \int d^n x \sqrt{-g} \left(\frac{1}{2} F^{\mu\nu} F_{\mu\nu} g_{\alpha\beta} - 2 F^{\mu}{}_{\alpha} F_{\mu\beta} \right) \delta g^{\alpha\beta} \,,
\end{aligned} \tag{71}$$

where we used $\frac{\delta g}{\delta g^{\alpha\beta}} = -g g_{\alpha\beta}$ and $\frac{\delta g^{\mu\nu}}{\delta g^{\alpha\beta}} = \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta}$. We read off the stress tensor

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\alpha\beta}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} g_{\alpha\beta} + F^{\mu}{}_{\alpha} F_{\mu\beta} \,, \tag{72}$$

as one obtains through Noether's theorem.