RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 12 Fall 2023

1. The Reissner-Nordström Black Hole

As argued in the textbook, for stationary metrics, if $g^{rr}(r) = 0$ in some coordinate system and for some specific value $r = r_H$, then there is a horizon at $r = r_H$. The horizon might not be apparent in some other coordinate system where g^{rr} is never vanishing.

(a) The horizon is thus where

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0\,. \tag{1}$$

This has solutions

$$r = M \pm \sqrt{M^2 - Q^2}\,. \tag{2}$$

Clearly, for $|Q| > M$ there are no real solutions, and thus there is no horizon. At the same time, there is a singularity at $r = 0$, proven by the fact that the scalar (and thus coordinate invariant) quantity $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is singular there. This means that this spacetime has a singularity that is not covered by an event horizon.

(b) The quantity that we are indicating as M here is really the total energy of the Black Hole, given by the sum of the rest masses of the particles that were thrown in to make the black hole plus the work done to add every new charged particle to the Black Hole. In order to have a total energy M that is lower than the total electric potential energy, we would need the rest masses of at least some of the particles to be negative.

(c) We know that this spacetime has a naked singularity and that it is asymptotically Minkowski (as we take $r \to \infty$). That means that the structure of spatial infinity should look exactly the same as in the Minkowski case. Moreover, since there is no place where the time coordinate becomes spacelike and the radial coordinate becomes timelike, the singularity is now a timelike line in the diagram. Look at Figure 6.3 on Sean Carroll's book.

(d) The metric for the extremal Black Hole reads

$$ds^2 = -\frac{(r - |Q|)^2}{r^2}\mathrm{d}t^2 + \frac{r^2}{(r - |Q|)^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2\,. \tag{3}$$

The horizon is now at $r = |Q|$. Performing the change of variabes $\varepsilon = r - |Q|$ we get

$$ds^2 = -\frac{\varepsilon^2}{(\varepsilon + Q)^2}\mathrm{d}t^2 + \frac{(\varepsilon + Q)^2}{\varepsilon^2}\mathrm{d}\varepsilon^2 + (\varepsilon + |Q|)^2\mathrm{d}\Omega_2^2\,. \tag{4}$$

To zoom in near the horizon, let us keep only the leading terms in ε

$$ds^2 = -\frac{\varepsilon^2}{Q^2}\mathrm{d}t^2 + \frac{Q^2}{\varepsilon^2}\mathrm{d}\varepsilon^2 + Q^2\mathrm{d}\Omega_2^2\,. \tag{5}$$

(e) Let us make one last change of variables to $\varepsilon = \frac{Q^2}{z}$. We obtain the following metric

$$ds^2 = Q^2 \left(\frac{-\mathrm{d}t^2 + \mathrm{d}z^2}{z^2} + \mathrm{d}\Omega_2^2 \right) . \tag{6}$$

The first term in the brackets is the Lorentzian analogue of the hyperbolic half plane metric we saw in Problem Set 7. This is AdS space. We can thus say that the near-horizon geometry of an extremal Black Hole is $AdS_2 \times S^2$, where S^2 is the two dimensional sphere in $\mathrm{d}\Omega_2^2$.

(f) As we computed before, we have two solutions for $M^2 > Q^2$

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \,, \tag{7}$$

indicating the presence of an outer and an inner event horizon. In both places, the coordinates t and r change nature from timelike to spacelike and viceversa. The singularity is thus timelike just like in the case where $Q^2 > M^2$.

(g) The area of the outer horizon, by spherical symmetry, is given by

$$A = 4\pi r_+^2 = 4\pi (2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2}) \,. \tag{8}$$

The differential is thus

$$dA = \frac{8\pi}{\sqrt{M^2 - Q^2}} \left(M + \sqrt{M^2 - Q^2} \right)^2 dM - 8\pi Q \left(1 + \frac{M}{\sqrt{M^2 - Q^2}} \right) dQ \,. \tag{9}$$

(h) Inverting this, we find

$$dM = \frac{\sqrt{M^2 - Q^2}}{8\pi (M + \sqrt{M^2 - Q^2})^2} dA + \frac{Q}{M + \sqrt{M^2 - Q^2}} dQ \,, \tag{10}$$

where we identify the surface gravity as

$$\kappa = \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2} = \frac{\sqrt{M^2 - Q^2}}{r_+^2} \,. \tag{11}$$

and the electric potential at the horizon

$$\Phi_H = -\frac{Q}{M + \sqrt{M^2 - Q^2}} = -\frac{Q}{r_+} \,. \tag{12}$$

That is the electric potential because $\Phi = \frac{\partial \mathcal{E}}{\partial Q}$ and the total energy of the system is $\mathcal{E} = M$.

(i) Given the differential form of the mass of the black hole, we can make an analogy with the first law of thermodynamics

$$\mathrm{d}U = \delta Q - \delta W \,, \tag{13}$$

where U is the internal energy of the system, δQ is the heat variation and δW is the mechanical work. For a reversible process, $\delta Q = T\mathrm{d}S$

$$\mathrm{d}U = T\mathrm{d}S - \delta W \,. \tag{14}$$

From the formula for the entropy of a black hole we have $\mathrm{d}A = 4\mathrm{d}S$. The formula for the variation of the mass is thus

$$dM = \frac{\kappa}{2\pi} dS - \Phi_H dQ \tag{15}$$

We thus identify the Temperature

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{r_+^2} = 2\frac{\sqrt{M^2 - Q^2}}{A} \,. \tag{16}$$

Notice that an extremal Black Hole, with $M = Q$, has zero temperature. Summarizing, the analogy with thermodynamics works as follows

$$U \leftrightarrow M \,, \qquad S \leftrightarrow \frac{A}{4} \,, \qquad T \leftrightarrow \frac{\kappa}{2\pi} \tag{17}$$

2. Penrose Diagram of de Sitter and Anti-de Sitter

(a) We are asked to show what is the explicit form of the metric on S^3. For general dimensions, angular coordinates ω^i are chosen such that the cartesian constraint for the coordinates on the sphere

$$\sum_{i=1}^{d} (\omega^i)^2 = 1 \tag{18}$$

is satisfied. The standard choice is to construct these coordinates by taking

$$\begin{aligned} \omega^1 &= \cos\theta_1 \,, \\ \omega^2 &= \sin\theta_1 \cos\theta_2 \,, \\ \omega^3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3 \,, \\ &\vdots \\ \omega^{d-1} &= \sin\theta_1 \cdots \sin\theta_{d-2} \cos\theta_{d-1} \,, \\ \omega^d &= \sin\theta_1 \cdots \sin\theta_{d-2} \sin\theta_{d-1} \,, \end{aligned} \tag{19}$$

with ranges $0 \le \theta_i < \pi$ for all angles except the last one $0 \le \theta_{d-1} < 2\pi$. You can check that (19) satisfy (18). Taking the differential and collecting the coefficients of each $\mathrm{d}\theta_i^2$ term, we get

$$\mathrm{d}\Omega_{d-1}^2 = \sum_{i=1}^{d} (\mathrm{d}\omega^i)^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2 + \ldots + \sin^2\theta_1 \cdots \sin^2\theta_{d-2} \mathrm{d}\theta_{d-1}^2 \,. \tag{20}$$

In particular, for S^3 we get

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta \; \mathrm{d}\phi^2 + \sin^2\theta \sin^2\phi \; \mathrm{d}\chi^2 \,. \tag{21}$$

(b) To find the metric in the new coordinates we compute the differential of $\cosh\tau = \frac{1}{\cos T}$

$$\begin{aligned} \sinh\tau \mathrm{d}\tau &= \frac{\tan T}{\cos T} \mathrm{d}T \\ \mathrm{d}\tau &= \frac{\tan T}{\cos T} \frac{1}{\sqrt{\cosh^2\tau - 1}} \\ \mathrm{d}\tau &= \frac{1}{\cos T} \,. \end{aligned} \tag{22}$$

The metric is thus

$$ds^2 = \frac{-\mathrm{d}T^2 + \mathrm{d}\Omega_3^2}{\cos^2 T}\,. \tag{23}$$

Once again we see that de Sitter is conformally related to the Einstein static universe. This time, though, the coordinates run independently, covering a stripe between $T = -\frac{\pi}{2}$ and $T = \frac{\pi}{2}$. Let us write the metric expliciting the dependency on one of the angles

$$ds^2 = \frac{1}{\cos^2 T}(-\mathrm{d}T^2 + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega_2^2)\,. \tag{24}$$

The ranges of values taken by θ and T are $\theta \in (0, \pi)$ and $T \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This differs from the case of Minkowski, where we found that there was an extra constraint $|T| < \pi - R$, leading to the famous triangle shape.

(c) Because that constraint $|T| < \pi - R$ is not there in de Sitter, the Penrose diagram will be shaped like a square. The left side of the square, at $\theta = 0$ is a point which we identify as the North pole of the S^3 sphere. The right side at $\theta = \pi$ is instead the South pole. Observers in these two places will never be able to interact with each other.

(d) By drawing light cones, it is straightforward to realize that there is no observer in de Sitter that has the full spacetime in its past. That means there can be multiple observers that never meet.

(e) The event horizon in de Sitter is an observer dependent concept. Consider an observer at rest on the South pole, without loss of generality. Draw her past light cone at every point in time. The union of all such light cones will be a right triangle with a diagonal that spans from the North pole at $\theta = 0$ to the South pole at $\theta = \pi$. This diagonal will be the observer's event horizon, because no signal from beyond that line will ever reach the observer.

(f) Under the suggested change in coordinates we get

$$\begin{aligned} \sinh\rho\ \mathrm{d}\rho &= \frac{\tan\chi}{\cos\chi}\ \mathrm{d}\chi \\ \mathrm{d}\rho &= \frac{\tan\chi}{\cos\chi}\frac{\mathrm{d}\chi}{\sqrt{\frac{1}{\cos^2\chi} - 1}} \\ &= \frac{\mathrm{d}\chi}{\cos\chi}\,. \end{aligned} \tag{25}$$

The metric is thus

$$ds^2 = \frac{1}{\cos^2\chi}(-\mathrm{d}t^2 + \mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\Omega_2^2)\,, \tag{26}$$

with $\chi \in (0, \frac{\pi}{2})$. The spacetime in the brackets is once again Einstein's static universe. This time, the coordinates cover half of it: the time coordinate runs from $-\infty$ to ∞, while the angular coordinate χ runs only from 0 to $\frac{\pi}{2}$ rather than to π.

(g) The Penrose diagram looks like an infinite rectangle. The left side is the origin at $\chi = 0$. Instead, $\chi = \frac{\pi}{2}$ corresponds to spatial infinity, $\rho \to \infty$, and in this spacetime it is timelike.

An extra bonus for the interested reader: the timelike nature of AdS's spatial infinity is the key to the AdS/CFT correspondance, a conjectured duality between quantum gravity in asymptotically n-dimensional AdS spacetimes and Conformal Field Theories (without dynamical gravity) that live on the $(n-1)$-dimensional cylinder at spatial infinity. The statement of the correspondance is that all observables computed at spatial infinity in quantum gravity in n-dimensional asymptotically AdS spacetimes give the same results as observables computed in specific CFTs on the n dimensional surface of the cylinder. While it has still not been proven, every case that can be checked respects this duality. The fact that the AdS/CFT correspondance relates theories of different dimensions, with gravity and without gravity, gives the name of **holography** to this duality.

(h) Acting with metric on $\frac{d}{ds}$ gives

$$-1 = \frac{1}{\cos^2\chi}\left(-\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{d\chi}{d\tau}\right)^2 + \dots\right) \tag{27}$$

As time is a cyclic coordinate, $\frac{dt}{\partial\tau} = E\cos^2\chi$. Transforming the equation

$$\left(\frac{d\chi}{d\tau}\right)^2 = E^2\cos^4\chi - \cos^2\chi \tag{28}$$

Using $\frac{d\chi}{d\tau} = \frac{d\chi}{dt}\frac{dt}{d\tau}$:

$$\left(\frac{d\chi}{dt}\right)^2 = 1 - \frac{1}{E^2\cos^2\chi} \tag{29}$$

This leads to differential equation

$$\left(\frac{d\chi}{dt}\right)^2 = 1 - \frac{1}{E^2\cos^2\chi} \tag{30}$$

Therefore, for every physical energy, $E > 1$, there exist a turning point $\cos\chi = 1/E$. To investigate further, let's integrate:

$$\pm\int \frac{E\cos\chi}{\sqrt{E^2\cos^2\chi - 1}}\mathrm{d}\chi = \int \mathrm{d}t \tag{31}$$

For range $\chi \in [0, \cos^{-1}(\frac{1}{E}))$,This results in[1]

$$\tan t = \frac{E\sin\chi}{\sqrt{E^2\cos^2\chi - 1}} \tag{32}$$

Which is an expression only valid due to ambiguity of trigonometric functions. Note that the turning point is $\left(t = \frac{\pi}{2}, \chi = \cos^{-1}(\frac{1}{E})\right)$ – and the t-coordinate of the point does not depend on E.

Assembling paths from origin to turning point and back gives surprising conclusion – every geodesics beginning in origin comes back to $\chi = 0$ after time $\Delta t = \pi$, and, as the space is maximally symmetric, this applies to every trajectory[2].

[1] Via substitution $u = \sin\chi$, $(1-u^2) = \cos^2\chi, \mathrm{d}u = \cos\chi\mathrm{d}\chi$

[2] Imagine throwing a stone and having it always hit your head after 3.14 s, regardless of your effort.

On the other hand, light-like geodesics follow the equations for $\mathrm{d}\tau = 0$, so

$$0 = \frac{1}{\cos^2\chi}\left(-\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{d\chi}{d\tau}\right)^2\right) \tag{33}$$

On range $t \in \left[0, \frac{\pi}{2}\right)$, they follow (not surprising, given the nature of procedure of creating a Penrose diagram) the path of

$$t = \chi$$

or, equivalently,

$$\tan t = \frac{\sin\chi}{\sqrt{\cos^2\chi}}$$

where one can see light-like geodesics is a limit of time-like with $E \lim \infty$. Careful reader may now backtrack to geodesics in Schwarzchild metric and other examples to see that's a common theme.

To complete the discussion, see the attached plots of (compactified) AdS geodesics with varying integration constant E: