RELATIVITY AND COSMOLOGY I

Problem Set 9

Fall 2023

1. Escaping the Photon Sphere

A photon is emitted from a point P outside a Schwarzschild black hole with radial coordinate r in the range $2M < r < 3M$. In this exercise we want to show that, in order for the photon to be able to escape the black hole, the angle its initial direction makes with the radial direction (as determined by a stationary observer located at P) has to satisfy the condition $\alpha \leq \alpha_0$ for some α_0 that we are going to compute.

(a) Compute the relation between energy E and angular momentum L for a photon orbiting a Schwarzschild black hole.[1] Study the effective potential and find the minimum value of E for the photon to be able to escape the black hole.

(b) By studying the components of the photon's velocity four-vector, express the angle α of emission with respect to the radial direction in terms of the photon's energy and angular momentum.

(c) Show that for the photon to escape the black hole,

$$\alpha \leq \arcsin\left[\sqrt{\frac{27M^2}{r^2}\left(1-\frac{2M}{r}\right)}\right]. \tag{1}$$

2. Orbiting Gargantua

Cooper and Amelia are at rest with radial coordinates r_C and r_A above the Gargantua black hole, with $r_C > 2M$ and $r_A > 2M$. Let τ_C and τ_A be their respective proper times.

(a) Suppose Cooper sends a repeating signal towards Amelia with period $\Delta\tau_C$. Show that she will measure

$$\Delta\tau_A = \frac{\sqrt{1-\frac{2M}{r_A}}}{\sqrt{1-\frac{2M}{r_C}}}\Delta\tau_C\,. \tag{2}$$

Now Cooper starts to free fall radially into Gargantua. We want to show that as he approaches the Event Horizon, Amelia stops receiving his signals, or in other words $\Delta\tau_A \to \infty$. In order to do that, we need to first argue that nothing special happens in Cooper's reference frame when he crosses the horizon, or in other words that $\Delta\tau_C$ remains finite.

(b) Using the conservation of energy, compute $\Delta\tau_C(E)$ as Cooper travels from r_C to $r_C - \Delta r_C$. Show that it is a finite quantity as $r_C \to 2M$.

[1] **Hint**: Argue that you can focus on orbits with $\theta = \frac{\pi}{2}$ without loss of generality.

(c) Compute the coordinate time interval Δt_C associated to the emission of two photons by Cooper as he crosses the event horizon, and the Δt_A associated to the reception of those photons by Amelia. Show that these quantities and $\Delta\tau_A$, in the $r_C \to 2M$ limit diverge.[2] What physical conclusion do you draw from this experiment?

3. Light Deflection in Scalar Gravity

During the lectures you have seen a scalar theory of gravitation that is relativistic and has the same Newtonian limit as GR called **Scalar Gravity**. You saw that this theory has timelike geodesics that satisfy

$$\frac{dU^\mu}{d\tau} = -\left(\eta^{\mu\nu} + U^\mu U^\nu\right)\partial_\nu \Phi\,. \tag{3}$$

(a) Consider a static potential $\partial_t \Phi = 0$ and solve the equations of motion in terms of the physical velocity. Show that particles that move close to the speed of light are not affected by the gravitational field.

(b) Show that the equations of motion (3) follow after putting the Lagrange multiplier $E(\lambda)$ on-shell[3] in the following action:

$$S = \frac{1}{2}\int \mathrm{d}\lambda \left[\frac{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{E(\lambda)} - m^2 e^{2\Phi(x(\lambda))}E(\lambda)\right]. \tag{4}$$

From this action it is manifest that for $m = 0$ the potential does not affect the geodesics.

[2]It is not possible to use (2) because Cooper is not at rest in our coordinate system anymore.

[3]Putting $E(\lambda)$ on-shell means solving the equations of motion for $E(\lambda)$ and substituting them in the action