RELATIVITY AND COSMOLOGY I

Problem Set 8 Fall 2023

1. Gravitational Redshift

In the presence of a weak static gravitational field ($\Phi \ll 1$), the metric is

$$ds^2 = -\left(1 + 2\Phi(x, y, z)\right) \mathrm{d}t^2 + \left(1 - 2\Phi(x, y, z)\right) \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{1}$$

In these coordinates, Alice is at position $\vec{x}_A$ and Bob at $\vec{x}_B$, and they are both static. Alice sends two photons towards Bob, one at t_1 and one at $t_1 + \Delta t_A$. Bob receives the first signal at T_1.

(a) Argue that in this setup, Bob receives the second signal at $T_1 + \Delta t_A$.

(b) What is the proper time interval $\Delta\tau_A$ that Alice measures between the two signals?

(c) What is the proper time interval $\Delta\tau_B$ that Bob measures between the two signals?

(d) Compare $\Delta\tau_B$ to $\Delta\tau_A$ in the weak field limit and give a physical interpretation to your result.

(e) Now imagine Alice generates an electromagnetic wave by oscillating a charge with a certain period T_A. What is the relation between the wavelength of the light as it is created by Alice and the wavelength of the light as it is measured by Bob? Give a physical interpretation to your answer.

(f) Argue that $K = \partial_t$ is a Killing vector of this metric.

(g) Show that, with $V^\mu = \frac{dX^\mu}{d\lambda}$, $\varepsilon = V_t$ is a constant of motion along the geodesics.

(h) As shown before, the energy of a particle, as measured by an observer, is $E = p_\mu U^\mu$, where U^μ is their 4-velocity. If Alice emits a photon of energy E_A, and this photon reaches Bob, what energy E_B will he measure? Compare to the previous answer.

2. The Bianchi Identity

In this exercise you are going to be guided through a proof of the **Bianchi identity**

$$\nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\nu R_{\rho\sigma\tau\mu} = 0\,, \tag{2}$$

where we are assuming a Levi-Civita connection.

(a) To start, show that (2) is equivalent to

$$\nabla_{[\tau} R_{\mu\nu]\rho\sigma} = 0\,. \tag{3}$$

(b) Show the following two identities for a generic dual vector V_ν.

$$[\nabla_\rho, \nabla_\sigma]\nabla_\mu V_\nu = -R^\lambda{}_{\mu\rho\sigma}\nabla_\lambda V_\nu - R^\lambda{}_{\nu\rho\sigma}\nabla_\mu V_\lambda\,, \tag{4}$$

$$\nabla_\rho[\nabla_\sigma, \nabla_\mu]V_\nu = -V_\lambda \nabla_\rho R^\lambda{}_{\nu\sigma\mu} - R^\lambda{}_{\nu\sigma\mu}\nabla_\rho V_\lambda\,. \tag{5}$$

(c) From the definition of the antisymmetrization alone, show that

$$\nabla_{[\rho}\nabla_\sigma\nabla_{\mu]}V_\nu - \nabla_{[\sigma}\nabla_\rho\nabla_{\mu]}V_\nu = \nabla_{[\rho}\nabla_\sigma\nabla_{\mu]}V_\nu - \nabla_{[\rho}\nabla_\mu\nabla_{\sigma]}V_\nu\,. \tag{6}$$

Argue that this is a version of **Jacobi's identity**.

(d) Use (4) on the left hand side and (5) on the right hand side of (6) to show that

$$-R^\lambda{}_{[\mu\nu\sigma]}\nabla_\lambda V_\nu - R^\lambda{}_{\nu[\rho\sigma}\nabla_{\mu]}V_\lambda = -V_\lambda\nabla_{[\rho}R^\lambda{}_{|\nu|\sigma\mu]} - R^\lambda{}_{\nu[\sigma\mu}\nabla_{\rho]}V_\lambda\,, \tag{7}$$

where notation like $A_{[\mu|\nu|\rho]}$ means the index ν is not being antisymmetrized over.

(e) Final step of the proof: show that (7) implies

$$\nabla_{[\tau} R_{\rho\sigma]\mu\nu} = 0\,, \tag{8}$$

which you argued is equivalent to the Bianchi identity.

You have shown the Bianchi identity. Now let us study one of its important consequences: the conservation of the Einstein tensor.

(f) By taking two traces of the Bianchi identity, show that

$$\nabla^\mu G_{\mu\nu} = 0\,, \tag{9}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ is the Einstein tensor.

3. The Coriolis Force

Consider a particle moving in Minkowski space with worldline $x^\mu(\lambda)$. We will denote $\frac{d}{d\lambda}$ with a dot. Consider the trajectories defined by $\ddot{x}^\mu = 0$.

(a) Show that this trajectory describes a particle moving with constant velocity.

(b) Show that this trajectory is a local extremum of the action

$$S = \int \mathrm{d}\lambda \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,. \tag{10}$$

Show that the action

$$S = \int \mathrm{d}\lambda \; \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\,. \tag{11}$$

leads to the same equations of motion if λ is an affine parameter.

(c) Consider a new coordinate system $x^{\mu'}$. Show that the equation of motion can be written in the new coordinate system as

$$\ddot{x}^{\mu'} + \Gamma^{\mu'}_{\nu'\lambda'}\dot{x}^{\nu'}\dot{x}^{\lambda'} = 0\,, \tag{12}$$

and compute the entries of $\Gamma^{\mu'}_{\nu'\lambda'}$. Notice that, in this new coordinate system, there seems to be a fictitious force acting on the particle.

(d) Consider the case in which the coordinate transformation relates two frames of reference that are rotating relative to each other

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t' \\ x'\cos(\omega t') + y'\sin(\omega t') \\ -x'\sin(\omega t') + y'\cos(\omega t') \\ z' \end{pmatrix}\,. \tag{13}$$

Compute the line element ds^2 in the new coordinate system.

(e) Compute the Christoffel symbols $\Gamma^{\mu'}_{\nu'\lambda'}$ in the new coordinate system. These terms describe the centrifugal and Coriolis forces that arise in a rotating coordinate system.