RELATIVITY AND COSMOLOGY I

Problem Set 7 Fall 2023

1. Hyperbolic Space

Maximally symmetric spaces with Euclidean signatures are planes, spheres or hyperboloids (and their higher dimensional counterparts). Consider a two dimensional hyperboloid $\mathbb{H}^2$ embedded in Minkowski $\mathbb{M}^3$, through the constraint[1]

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = -L^2\,, \qquad X^0 > 0\,, \tag{1}$$

where X^μ are the coordinates in $\mathbb{M}^3$.

(a) Verify that two possible choices of coordinates on $\mathbb{H}^2$ that solve this constraint are

$$\begin{cases} X^0 = L\frac{1+x^2+z^2}{2z}\,, \\ X^1 = L\frac{x}{z}\,, \\ X^2 = L\frac{1-x^2-z^2}{2z}\,. \end{cases} \,, \qquad \begin{cases} X^0 = \frac{L}{\cos\rho}\,, \\ X^1 = L\cos\theta\tan\rho\,, \\ X^2 = L\sin\theta\tan\rho\,. \end{cases} \tag{2}$$

(b) Show that the metric in these two coordinate systems is

$$\begin{aligned} ds^2 &= L^2\frac{\mathrm{d}\rho^2 + \sin^2\rho\ \mathrm{d}\theta^2}{\cos^2\rho}\,, \qquad \rho \in [0, \frac{\pi}{2}) \text{ and } \theta \in [0, 2\pi) \\ ds^2 &= L^2\frac{\mathrm{d}x^2 + \mathrm{d}z^2}{z^2}\,, \qquad x \in \mathbb{R} \text{ and } z \in (0, \infty)\,. \end{aligned} \tag{3}$$

The first one defines the Poincaré disk model, of which you probably saw artistic renditions by *M.C. Escher*. In those coordinates, the total hyperbolic space is compactified to a disk. The second coordinate system is called **Poincaré coordinates**, or the Poincaré half-plane.

(c) Show that the length of a segment on the half plane with fixed x that starts at z_1 and ends at z_2 is

$$\Delta s = L\ln\left(\frac{z_2}{z_1}\right)\,. \tag{4}$$

What does this mean for straight paths on the (x, z) plane that reach the boundary at $z = 0$?

(d) Let's parameterize a curve using a coordinate $x^{\bar{\sigma}}$ instead of an affine parameter λ. Using the chain rule $\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^{\bar{\sigma}}}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}x^{\bar{\sigma}}}$, show that the geodesic equation for such a curve is

$$\frac{\mathrm{d}v^\rho}{\mathrm{d}x^{\bar{\sigma}}} = -\Gamma^\rho_{\mu\nu}v^\mu v^\nu + v^\rho\Gamma^{\bar{\sigma}}_{\mu\nu}v^\mu v^\nu\,, \tag{5}$$

where $v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}x^{\bar{\sigma}}}$. Write this equation for $z(x)$ and find the geodesics on the Poincaré half-plane. You should find that geodesics are straight lines perpendicular to the boundary and semicircles centered on the boundary.

[1] Euclidean hyperbolic spaces of dimension n can only be isometrically embedded in $\mathbb{R}^{n+2}$ or in $\mathbb{M}^{n+1}$.

(e) Find the Killing vectors of hyperbolic space. Show that they can be written as linear combinations of the following vectors

$$P = \partial_x\,, \qquad D = x\partial_x + z\partial_z\,, \qquad K = \left(x^2 - z^2\right)\partial_x + 2xz\partial_z\,. \tag{6}$$

What is their algebra under commutation?[2]

2. Lie Derivatives

In this exercise we are going to introduce the concept of **Lie derivative** - a differential operator that satisfies the Leibniz rule: $\mathcal{L}_V(AB) = A(\mathcal{L}_V B) + (\mathcal{L}_V A)B$. Consider vectors V and W, dual vectors S and T and a scalar function f. The Lie derivative is defined as

$$\mathcal{L}_V f \equiv V^\mu \partial_\mu f\,, \qquad (\mathcal{L}_V W)^\nu \equiv V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu\,, \tag{7}$$

where $\mathcal{L}_A B$ means *Lie derivative of B with respect to A*. For a generic tensor T, the Lie derivative $\mathcal{L}_V T$ essentially codifies the change of T along V.

(a) Show that

$$(\mathcal{L}_V W)^\alpha = -(\mathcal{L}_W V)^\alpha\,. \tag{8}$$

(b) Show that $(\mathcal{L}_V W)^\alpha$ transform like the components of a vector under coordinate transformations.

(c) Show that[3]

$$(\mathcal{L}_V T)_\alpha = V^\mu \partial_\mu T_\alpha + T_\mu \partial_\alpha V^\mu\,. \tag{9}$$

(d) Consider a tensor $A_{\mu\nu} = S_\mu T_\nu$. Show that

$$(\mathcal{L}_V A)_{\mu\nu} = V^\rho \partial_\rho A_{\mu\nu} + A_{\rho\nu}\partial_\mu V^\rho + A_{\mu\rho}\partial_\nu V^\rho\,. \tag{10}$$

(e) Assume that you have a metric $g_{\mu\nu}$ and that ∇_μ is a covariant derivative with respcet to the Levi-Civita connection. Show that in (7) and (9) you can equivalently write ∂_μ or ∇_μ.

(f) Show that Killing's equation

$$\nabla_{(\mu}K_{\nu)} = 0\,, \tag{11}$$

is equivalent to

$$(\mathcal{L}_K g)_{\mu\nu} = 0 \tag{12}$$

A short comment: the Lie derivative of a vector field W with respect to another vector field V is also referred to as the Lie bracket between the two vectors, and is essentially given by the commutator $[V, W]$. The space of vector fields forms a Lie algebra with respect to the Lie bracket.

[2]**Hint**: A maximally symmetric n-dimensional space has $\frac{n}{2}(n+1)$ symmetries.
[3]**Hint**: Form a scalar with T_μ and some auxiliary vector.

3. Killing Vectors on the Sphere

(a) Find the Killing vectors of S^2 with metric[4]

$$ds^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \,. \tag{13}$$

(b) Verify that the Killing vectors you found satisfy an algebra with respect to the Lie bracket that is isomorphic to $\mathfrak{so}(3)$.

4. Maxwell's Stress Tensor

Consider the minimally coupled Maxwell action with no sources

$$S_M[g_{\mu\nu}, A_\mu] = \int d^n x \sqrt{-g} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \,. \tag{14}$$

Use functional differentiation to compute the associated stress tensor,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \,. \tag{15}$$

[4]**Hint**: Embed the sphere in $\mathbb{R}^3$ and consider rotations.