RELATIVITY AND COSMOLOGY I

Problem Set 6 Fall 2023

1. The Expanding Universe

At the largest scales, our universe is in first approximation uniform, homogeneous and isotropic in space. The line element on such scales can be written as

$$ds^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) , \tag{1}$$

where $a(t)$ is called the scale factor, and it sets the rate of expansion of the universe. Metrics with such a form are called **Friedmann-Robertson-Walker** (FRW) metrics.

(a) Show that the non-vanishing Christoffel symbols for this metric are

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij} , \qquad \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j , \tag{2}$$

where dots are derivatives with respect to coordinate time t.

The energy of a particle with momentum p^μ as measured by an observer with four-velocity U^μ_{obs} is given by

$$E = -p_\mu U^\mu_{\text{obs}} . \tag{3}$$

For a massive particle, $p^\mu = m\frac{dx^\mu}{d\tau}$. For a photon, proper time vanishes and there is no unique parametrization of its geodesics. The requirement $p^\mu = \frac{dx^\mu}{d\lambda}$ fixes the normalization of λ and once an initial condition is chosen, the geodesic equation uniquely defines the rest of the curve.

(b) Compute the energy of a photon in an expanding universe as measured by an observer at rest.[1]

We can model the matter and energy contents of the universe as a homogeneous fluid. The stress energy tensor will thus be given by

$$T^{\mu\nu} = (\rho + P)U^\mu_{\text{fluid}}U^\nu_{\text{fluid}} + Pg^{\mu\nu} , \tag{4}$$

where ρ and P are respectively the energy density and the pressure of the fluid.

(c) Show that conservation of energy and momentum $\nabla_\mu T^{\mu\nu} = 0$ in the rest frame of the fluid implies

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) , \qquad \partial_i P = 0 . \tag{5}$$

(d) Consider an equation of state of the form $P = \omega\rho$ with ω being a constant. Show that in that case

$$\rho \propto a^{-3(1+\omega)} . \tag{6}$$

Can you interpret the physics of your results in the cases $\omega = 0$, $\omega = \frac{1}{3}$ and $\omega = -1$?

[1] Hint: start from the condition that its path be null $ds^2 = 0$.

2. Geodesic Deviation

In flat space, parallel geodesics stay parallel. In a curved space this is not true, and the Riemann tensor has the role of quantifying the geodesic deviation from parallelism. Consider a one-parameter family of neighboring geodesics $\gamma_s(t)$, where t is their affine parameter and each value of $s \in \mathbb{R}$ corresponds to a different geodesic. This family of geodesics defines a surface $x^\mu(s,t)$. It is natural to consider the tangent vectors T^μ and the deviation vectors S^μ defined through

$$T^\mu = \frac{\partial x^\mu}{\partial t}\,, \qquad S^\mu = \frac{\partial x^\mu}{\partial s}\,. \tag{7}$$

We want to quantify the deviation from parallelism, or in other words the tendency of two geodesics to grow apart as we move along t. It makes sense to define for that purpose the **relative velocity of geodesics**

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu\,, \tag{8}$$

and the **relative acceleration of geodesics**

$$A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu\,. \tag{9}$$

(a) Argue that $[T,S]=0$.

(b) Use this to show that[2]

$$A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma\,. \tag{10}$$

This is the **geodesic deviation equation**.

3. The Riemann Tensor

Using that

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}\,, \qquad R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}\,, \qquad R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0\,, \tag{11}$$

show that[3]

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}\,. \tag{12}$$

[2]**Hint**: You need to add and subtract some terms such that $[\nabla_\rho, \nabla_\sigma]T^\mu = R^\mu{}_{\nu\rho\sigma}T^\nu$ appears.

[3]**Hint**: Start from $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$ and its permutations.

4. The Curvature of S^3

Consider the metric for S^3 in (ψ, θ, ϕ) coordinates

$$ds^2 = r^2 \mathrm{d}\psi^2 + r^2 \sin^2\psi (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,, \tag{13}$$

where r is the radius of the sphere. In what follows, we will set $r = 1$ for simplicity.

(a) Compute the Christoffel coefficients of this metric by imposing that the variation of

$$I = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda \tag{14}$$

with respect to each coordinate should vanish. Why is this a sensible thing to do?

(b) Compute the components of the Riemann tensor, the Ricci tensor and the Ricci scalar. [4]

(c) Verify that

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})\,, \tag{15}$$

as expected from any maximally symmetric space.

(d) Argue that all the components of the Weyl tensor

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)}g_{\rho[\mu}g_{\nu]\sigma}R\,. \tag{16}$$

should vanish in this space.

[4] **Hint**:Use the symmetries from Problem 3 to reduce the amount of computations needed. Remember that the Riemann tensor has $\frac{n^2}{12}(n^2-1)$ independent components, where n is the dimension of the manifold.