
RELATIVITY AND COSMOLOGY I

Problem Set 5

Fall 2023

1. Christoffel Symbols for a Diagonal Metric

Consider the formula for Christoffel symbols

$$\Gamma_{\mu\nu}^{\sigma} = \frac{1}{2} g^{\sigma\rho} (\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu}) . \quad (1)$$

(a) Show that, for a diagonal metric,

$$\begin{aligned} \Gamma_{\bar{\mu}\bar{\nu}}^{\bar{\lambda}} &= 0 , & \Gamma_{\bar{\mu}\bar{\mu}}^{\bar{\lambda}} &= -\frac{1}{2g_{\bar{\lambda}\bar{\lambda}}} \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\mu}} , \\ \Gamma_{\bar{\mu}\bar{\lambda}}^{\bar{\lambda}} &= \partial_{\bar{\mu}} \left(\ln \sqrt{|g_{\bar{\lambda}\bar{\lambda}}|} \right) , & \Gamma_{\bar{\lambda}\bar{\lambda}}^{\bar{\lambda}} &= \partial_{\bar{\lambda}} \left(\ln \sqrt{|g_{\bar{\lambda}\bar{\lambda}}|} \right) , \end{aligned} \quad (2)$$

where $\bar{\lambda} \neq \bar{\mu} \neq \bar{\nu}$ and no Einstein summation is implied in all these equations.

These will be useful formulas to quickly compute the Christoffel symbols of diagonal metrics.

2. Geodesics

(a) An affine parameter is any λ that is linearly related to proper time $\lambda = a\tau + b$. Geodesics $x^{\mu}(\lambda)$ parametrized by affine parameters satisfy the equation

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma_{\rho\sigma}^{\mu} \frac{dx^{\rho}}{d\lambda} \frac{dx^{\sigma}}{d\lambda} = 0 . \quad (3)$$

Show that, for a non-affine parameter $\alpha = \alpha(\lambda)$, the geodesic equation becomes

$$\frac{d^2 x^{\mu}}{d\alpha^2} + \Gamma_{\rho\sigma}^{\mu} \frac{dx^{\rho}}{d\alpha} \frac{dx^{\sigma}}{d\alpha} = f(\alpha) \frac{dx^{\mu}}{d\alpha} , \quad (4)$$

and find $f(\alpha(\lambda))$.

(b) Consider a geodesic $x^{\mu}(\lambda)$ parametrized by an affine parameter λ . Call $V = V^{\mu} \partial_{\mu}$ the tangent vector to the geodesic. Show that

$$V^{\mu} \nabla_{\mu} V^{\nu} = 0 \quad (5)$$

is the geodesic equation.

3. Geodesics on S^2

Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 . \quad (6)$$

- (a) Write down the geodesic equation in this coordinate system using affine parametrization.
- (b) Show that lines of constant longitude ϕ are geodesics, and that the only line of constant latitude that is a geodesic is the equator ($\theta = \frac{\pi}{2}$).
- (c) Take a vector with components $V^\mu = (1, 0)$ and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector, as a function of the latitude θ ?

4. Divergence and Laplacian

Show that the following formulas hold for a generic metric:

- (a) Divergence of a vector

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} V^\mu \right) . \quad (7)$$

- (b) Laplacian on a scalar

$$\nabla_\mu \nabla^\mu f = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu f \right) . \quad (8)$$