
RELATIVITY AND COSMOLOGY I

Problem Set 14

Fall 2022

1. The Hulse–Taylor Binary

The Hulse–Taylor binary is a binary star system composed of a neutron star and a pulsar with nearly equal mass $M \sim 1.4M_{\text{Sun}}$ which are on an elliptical orbit with their common center of mass in one of the foci. It is the first binary pulsar ever discovered, which earned the two scientists a Nobel prize in 1993. The most interesting feature of this system is that the orbital period can be measured with high precision, and it shows the expected decay caused by the emission of gravitational waves.

In the lectures you saw that an isolated nonrelativistic object produces a gravitational wave given by the **quadrupole formula**

$$\bar{h}_{ij}(t, \vec{x}) = \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}(t_r), \quad I_{ij}(t) = \int y^i y^j T^{00}(t, \vec{y}) d^3 y, \quad t_r \equiv t - r. \quad (1)$$

- (a) Parametrize the orbit of the two stars. As a simplifying assumption, keep their orbit circular.
- (b) Write down the energy density $T^{00}(t, \vec{y})$ for this system.
- (c) Derive the quadrupole moment $I_{ij}(t)$.

The power P radiated by a gravitational system depends on the traceless part of the quadrupole moment

$$J_{ij} \equiv I_{ij} - \frac{1}{3} \delta_{ij} \delta^{kl} I_{kl}, \quad (2)$$

and takes the form

$$P = -\frac{G}{5} \left\langle \frac{d^3 J_{ij}}{dt^3} \frac{d^3 J^{ij}}{dt^3} \right\rangle, \quad (3)$$

where the angle brackets indicate an average over time.

- (d) Derive the power emitted by the binary system.
- (e) How does the rotational period of the binary system change as a function of time? Treat their orbit with Newtonian gravity.
- (f) What is the order of magnitude of the lifetime of this system, knowing that the current orbital period is 7.75 hr?

2. The Inside of a Fermi Gas Star

In this exercise, we are going to explore an idealized model of the interior of a star in General Relativity. Generically, star interiors are described by the line element

$$ds^2 = -e^{2\alpha(r)} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \quad r < R. \quad (4)$$

The total mass between radius values 0 and r is given by

$$m(r) = \int_0^r 4\pi r'^2 \rho(r') dr', \quad m(0) = 0. \quad (5)$$

The **Tolman-Oppenheimer-Volkoff** (TOV) equation of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r(r - 2m)}, \quad P(0) = P_c \quad (6)$$

and $\alpha(r)$ satisfies

$$\frac{d\alpha}{dr} = \frac{m + 4\pi r^3 P}{r(r - 2m)}, \quad \alpha(r = R) = \frac{1}{2} \ln \left(1 - \frac{2M}{R}\right). \quad (7)$$

To construct a stellar model, one simply specifies an equation of state for ρ and P , a value of the central pressure P_c and an arbitrary value $\alpha_0 = \alpha(0)$. Then, one integrates these equations outward until $P = 0$. The point at which $P = 0$ is by definition the radius of the star R . There are few idealized examples in which this can be done analytically. In class, you saw the example of a star made of an incompressible fluid. Here, we explore the idealized case of a star made of an ideal Fermi gas at zero temperature, so highly compressed that the particles have relativistic energies. In this limit, their rest masses are negligible, and the equation of state is $P = \frac{\rho}{3}$.

- (a) Write down the TOV equation for this system,
- (b) Use the ansatz $m(r) = Ar$ and find the constant A .
- (c) What are the central density and pressure for this solution?
- (d) Find the metric inside the star.
- (e) Concentrate on the spatial part of the metric. Show that it presents a conical singularity at $r = 0$. What is the unphysical aspect of this model that causes this artifact in the geometry?
- (f) You should be able to see that the radius of this star is infinite. This is due to another flaw in this idealized model. Can you identify it?