RELATIVITY AND COSMOLOGY I

Problem Set 13 Fall 2022

1. Diffeomorphism invariance and the Conservation of $T^{\mu\nu}$

Diffeomorphisms are smooth and differentiable coordinate transformations.

(a) Show that, under an infinitesimal diffeomorphism $x^\mu \to x^\mu - \xi^\mu$ with $\xi^\mu \ll x^\mu$, the metric transforms as

$$\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}\,. \tag{1}$$

(b) Show that theories with diffeomorphism invariance have a covariantly conserved stress energy tensor.[1]

2. Frame Dragging

Consider a thin spherical shell of matter, with total mass M and radius R, slowly rotating[2] with angular velocity Ω. We are going to be interested in what happens inside this shell.

(a) Write the components of the stress energy tensor in spherical polar coordinates for this setup.

In the Lorenz gauge, the linearized Einstein equations look like

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}\,, \tag{2}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ is the trace-reversed metric perturbation.

(b) Argue that the T_{ij} components can be neglected in the limit of small velocity.

(c) Given that (2) for $\bar{h}_{0\mu}$ looks like Maxwell's equations in Lorenz gauge, $\Box A_\mu = -J_\mu$, identify the gravito-electric ($\vec{G}$) and the gravito-magnetic ($\vec{H}$) fields.

(d) Show that the spatial components of the geodesic equation, in this setup, are analogous to the Lorentz force in electromagnetism

$$\frac{dv^i}{dt} = G^i + (\vec{v}\times H)^i + O(v^2)\,. \tag{3}$$

(e) Compute the gravito-magnetic field[3] $\vec{H}$ inside the sphere in terms of M, R and Ω. Argue that the gravito-electric field vanishes there.

(f) The nonzero gravito-magnetic field caused by the shell "drags" inertial frames, giving rise to what is called the **Lense-Thirring effect**. To see this, calculate the precession of the spatial components of a parallel-transported vector located at the center. What is the physical effect on a free-falling observer at the center of the sphere?

[1] **Hint**: impose that the action is invariant under a diffeomorphism and integrate by parts.
[2] Keep only terms that are linear in the velocity
[3] **Hint**: Use the multipole expansion. Argue that only the harmonic with $\ell = 1$ contributes.

3. Gravitational Plane Waves

In this exercise we are going to derive various properties of gravitational plane waves. Throughout it, you can use the linearized formula for Christoffel symbols

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu})\,, \tag{4}$$

and the one for the Riemann tensor

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}) \tag{5}$$

We are going to work in the transverse traceless (TT) gauge, defined by the gauge conditions

$$h^{\mathrm{TT}}_{0\nu} = 0\,, \qquad \eta^{\mu\nu}h^{\mathrm{TT}}_{\mu\nu} = 0\,, \qquad \partial^\mu h^{\mathrm{TT}}_{\mu\nu} = 0\,, \tag{6}$$

which fix, together with the Einstein equations,

$$h^{\mathrm{TT}}_{\mu\nu} = C_{\mu\nu}e^{ik_\sigma x^\sigma}\,, \qquad C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{7}$$

where we should remember to take the real part at the end of our calculations.

(a) Show explicitly that, in the TT gauge, particles at rest feel no acceleration from the passage of a gravitational wave. [4]

To appreciate the effect of a gravitational wave we need to look at the relative motion of a set of particles. Consider a coordinate system centered around the geodesic of one of these particles. The separation of the other particles from the origin evolves in time following the geodesic deviation equation

$$\frac{D^2 S^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma}U^\nu U^\rho S^\sigma\,, \tag{8}$$

(b) Consider as initial conditions, a setup in which all the particles are at rest relative to each other and compute the relevant entries of the Riemann tensor.

(c) Argue that, in this setup,

$$\frac{D^2 S^\mu}{d\tau^2} = \frac{\partial^2 S^\mu}{\partial t^2} \tag{9}$$

(d) Consider a right-handed circularly polarized wave, given by the condition $h_\times = -ih_+ \equiv h_R$ and integrate (8) at first order to find the components of the separation vector S^μ as a function of time if the wave propagates in the x^3 direction. Can you give a physical interpretation to your solution?[5]

[4] **Hint**: Start from the geodesic equation and impose the relevant initial conditions.

[5] **Hint**: Use Mathematica to solve this differential equation or use the ansatz $S^i(t) = a + bt + c\cos(dt) + e\sin(ft)$.