RELATIVITY AND COSMOLOGY I

Problem Set 11 Fall 2022

1. The Penrose Diagram of Minkowski

Penrose diagrams allow us to compactly visualize the entirety of a spacetime while keeping its causal structure manifest. In order to achieve that, they are derived by transforming your starting metric to coordinates that have finite ranges and in which light rays propagate at 45 degrees. In this exercise you will be guided towards a derivation of the Penrose diagram of Minkowski space.

(a) Consider Minkowski space in spherical coordinates

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \,. \tag{1}$$

A naive guess for some good compact coordinates would be

$$\bar{T} = \arctan t \,, \qquad \bar{R} = \arctan r \,. \tag{2}$$

Why is this a **bad** choice to build our Penrose diagram?

(b) Instead, start by performing the following coordinate transformation

$$u = t - r \,, \qquad v = t + r \,. \tag{3}$$

Write down the metric of Minkowski space in these coordinates. What are their ranges?

(c) Now go to the so-called Penrose coordinates

$$U = \arctan u \,, \qquad V = \arctan v \,. \tag{4}$$

What is the range of U and V? Write down the metric again.[1]

(d) Finally, go to the coordinates

$$T = V + U \,, \qquad R = V - U \,. \tag{5}$$

What is their range? You should be able to write the metric in these coordinates as

$$ds^2 = \omega(T, R) \left[-\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R \mathrm{d}\Omega^2 \right] \,, \tag{6}$$

for some function $\omega(T, R)$ you have to find. What is the manifold described by the metric in the square brackets? How do radial light rays propagate in these coordinates?

(e) Represent on a diagram the spacetime in the square brackets in (6). The ranges of R and T are important. Draw surfaces of constant r and constant t. How do null geodesics look like on this diagram?

[1] **Hint**:It is going to simplify the following steps if you realize that the coefficient of $\mathrm{d}\Omega^2$ in these coordinates can be written as $\frac{\sin^2(V-U)}{f(U,V)}$, with $f(U, V)$ some function you will calculate.

2. A Hamiltonian Approach to Geodesics

During the lectures you've seen that geodesics can be found by extremizing the following action:

$$S = \frac{1}{2}\int \left(\frac{1}{\xi} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - \xi m^2\right) \mathrm{d}\lambda = \int \mathcal{L}(x^\mu, \dot{x}^\mu, \xi) \mathrm{d}\lambda\,,$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$. In classical mechanics, a system can be equivalently described by a Lagrangian (in terms of positions and velocities) or a Hamiltonian (function of positions and momenta). In this problem you will investigate conveniences and inconveniences of the Hamiltonian approach to the geodesic equation.

(a) The momentum p_μ conjugate to the variable x_μ is given by $p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$. Compute the **Legendre transform** of $\mathcal{L}(x, \dot{x}, \xi)$:

$$\mathcal{H}(x, p, \xi) = p_\mu \dot{x}^\mu - \mathcal{L}$$

Note: The transformation in ξ is trivial, as $p_\xi = 0$

(b) Show that the Hamilton equations of motion $\dot{x}^\mu = \frac{\partial \mathcal{H}}{\partial p_\mu}$, $\dot{p}_\mu = -\frac{\partial \mathcal{H}}{\partial x^\mu}$ are equivalent to the geodesic equations.

(c) Find the equations of motion for ξ. Prove that, for consistency, the value of $\mathcal{H}$ must be 0 on-shell. What are the constraints imposed on the momenta? How do they translate to constraints on the velocities?

(d) Prove that, if the metric is not a function of the coordinate $x^{\bar{\mu}}$, $p_{\bar{\mu}}$ is a conserved quantity.

(e) Let us introduce the Poisson brackets:

$$\{a, b\} = \frac{\partial a}{\partial x^\mu}\frac{\partial b}{\partial p_\mu} - \frac{\partial a}{\partial p_\mu}\frac{\partial b}{\partial x^\mu}$$

Show that, for a function $f(x, p)$, the equations of motion imply $\dot{f} = \{f, \mathcal{H}\}$.

(f) Consider a vector field $K^\mu = K^\mu(x)$ and the quantity $f(x, p) = p_\mu K^\mu$. Show that this is a conserved quantity if and only if K^μ is a Killing vector.

3. The vielbein formalism

In this exercise we will introduce a different formalism for general relativity that can simplify many computations.

(a) Argue that the metric $g_{\mu\nu}(x)$ can be written

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}, \tag{7}$$

where η_{ab} is the Minkowski metric. (*Hint :* symmetric matrices can be diagonalized.) The fields $e^\mu_a(x)$ are called **vielbeins** or **frame fields**. The indices μ, ν are the usual spacetime indices and the indices a, b are called the frame indices. We define their inverses by switching the place of the indices as

$$e^\mu_a e^a_\nu = \delta^\mu_\nu, \qquad e^a_\mu e^\mu_b = \delta^a_b. \tag{8}$$

(b) Show that the vielbeins are not uniquely defined. There is a transformation

$$e'^a_\mu(x) = T^a_{\ b}(x) e^b_\mu(x) \tag{9}$$

leaving their defining equation invariant. What is $T^a_{\ b}$?

(c) We can think of the fields e^a_μ as the components of a 1-form e^a. Show that

$$ds^2 = e^a e^b \eta_{ab} \equiv e^a e_a, . \tag{10}$$

(d) We can define frame vector components as $V^a \equiv e^a_\mu V^\mu$. The covariant derivative in that basis is written

$$\nabla_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b, \tag{11}$$

where $\omega_\mu{}^a{}_b$ is called the **spin connection**. Show that

$$\Gamma^\nu_{\mu\lambda} = e^\nu_a \partial_\mu e^a_\lambda + e^\nu_a e^b_\lambda \omega_\mu{}^a{}_b, \qquad \omega_\mu{}^a{}_b = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda. \tag{12}$$

(e) Defining the connection 1-form $\omega^a{}_b \equiv \omega_\mu{}^a{}_b dx^\mu$, first show that, using $\nabla_\mu \eta_{ab} = 0$, ω_{ab} are antisymmetric in a, b. Then show that

$$de^a + \omega^a{}_b \wedge e^b = e^a_\rho e^a_\lambda \Gamma^\lambda_{\nu\mu} dx^\mu \wedge dx^\nu \tag{13}$$

If we use the Levi-Civita connection we get

$$de^a + \omega^a{}_b \wedge e^b = 0. \tag{14}$$

This is known as the first structure equation. Now we define the curvature 2-form $R^a{}_b \equiv e^a_\lambda e^\rho_b R^\lambda{}_{\rho\mu\nu} dx^\mu \wedge dx^\nu$, show that

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{15}$$

which is known as the second structure equation.

(f) We will now see how to use all the formalism we just developed. Consider the FLRW metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{16}$$

Use (10) to compute the vielbeins e^a_μ of that metric. Then use the first structure equation (14) to compute the components of the spin connection $\omega^a{}_b$ (remember that ω_{ab} is antisymmetric in a, b). Then use the second structure equation (15) to compute the Riemann tensor and write down the Einstein's equations with the stress-energy tensor of a perfect fluid. You will get the Friedmann equations.