Relativity and Cosmology I

Practice exam - Fall 2022

Some advice:

- You can use a formulaire (max 4 pages and readable without magnifying glasses).
- You may use formulas explained in the lectures without deriving them, as long as you make your reasoning clear.
- Make sure your final result is not absurd (we subtract up to 50% of the mark in this case). A final result can be absurd for several reasons: it is dimensionally incorrect, it is incompatible with an obvious symmetry or conservation law, it goes against basic physical intuition, etc.
- Write your answers with a pen and with clear handwriting.

4. Scalar field

Consider a minimally coupled scalar field ϕ in curved spacetime

$$S = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + U(\phi)\right) ,$$

with $U(\phi)$ an arbitrary potential.

(a) Compute the equation of motion and the energy-momentum tensor for the field.

(b) Assuming ϕ to be *homogeneous*, write down its energy-momentum tensor and compare it to the one of a perfect fluid. What is the equation of state in the limit $\left(\frac{d\phi}{dt}\right)^2 \ll U(\phi)$? Give a physical interpretation to your answer.

5. Lemaître coordinates

The Schwarzschild metric can be written as

$$ds^2 = -\left(1 - \frac{r_S}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{r_S}{r}} + r^2 d\Omega^2 . \tag{8}$$

Consider an initially stationary body, falling into the black hole from large distance (asymptotically far).

(a) Show that the 4-velocity of this observer is

$$U = U^t\partial_t + U^r\partial_r \tag{9}$$

and determine the functions U^t and U^r.
Hint: notice that ∂_t is a Killing vector.

(b) Find a vector V perpendicular to U so that

$$[U, V] = 0 . \tag{10}$$

Hint: use the ansatz $V = V^t(r)\partial_t + V^r(r)\partial_r$.

(c) We can associated a new coordinate system to the commuting vector fields U, V. Write $U = \partial_\tau$ and $V = \partial_\rho$ and find the dual 1-forms $d\tau, d\rho$. [1] Write the metric in terms of these 1-forms.

(d) Find r as a function of ρ, τ, so that $r = 0$ corresponds to $\rho = \tau$.

(e) Depict the causal structure on the (ρ, τ) plane. In particular, draw the singularity and the event horizon.

(f) In new coordinates, show the trajectory $\rho = 0$ is a geodesic. What is proper time taken by a body following this geodesic to go from the horizon to the singularity?

[1] Recall that $d\tau(\partial_\tau) = 1$, $d\rho(\partial_\rho) = 1$, $d\tau(\partial_\rho) = 0$ and $d\rho(\partial_\tau) = 0$.

(g) What is the minimal and maximal proper time that an unaccelerated body can take to go from the horizon to the singularity?

6. Extra dimension

Consider a 5-dimensional spacetime with metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + R^2 \left(d\phi + A_\mu(x) dx^\mu\right)^2 \tag{11}$$

where $\mu, \nu \in \{0, 1, 2, 3\}$ and ϕ is a periodic coordinate with period 2π.

(a) What is the effect on $A_\mu(x)$ of a coordinate transformation $\phi \to \phi + f(x)$?

(b) What is the Einstein-Hilbert action evaluated on this metric? Make an educated guess to avoid a lengthy explicit computation.