Relativity and Cosmology I

Written exam - 29/01/2024

Some advice:

- You can use a formulaire (max 4 pages and readable without magnifying glasses).
- You may use formulas explained in the lectures without deriving them, as long as you make your reasoning clear.
- Make sure your final result is not absurd (we subtract up to 50% of the mark in this case). A final result can be absurd for several reasons: it is dimensionally incorrect, it is incompatible with an obvious symmetry or conservation law, it goes against basic physical intuition, etc.
- Write your answers with a pen and with clear handwriting.

1 Time delays

Consider the gravitational field of Earth

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1}$$

where $\Phi = -GM/r$, with M the mass of the Earth. You may work to first order in the gravitational potential $\Phi \ll 1$.

1. Imagine a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock ticks faster?

2. Consider another clock on a satellite (without rockets) following a circular orbit of radius R_3 on the equatorial plane $\theta = \pi/2$. Calculate the time elapsed on this clock as a function of the coordinate time t.

3. For what orbital radius R_3 the clock on the satellite remains synchronised with the clock on the surface of the Earth?

4. GPS satellites orbit at an altitude of about 20000 km from the ground. Estimate the time difference, produced per day, between the clock on a GPS satellite and a ground clock.

   $GM/c^2 = 4.4 \times 10^{-3}\, m$
   $R_1 = 6.6 \times 10^6\, m$

2 Period change

Consider a planet of mass m in a circular orbit of period T around a star of mass $M \gg m$. In this problem, you can use Newtonian mechanics to determine the trajectory of the planet.

1. Compute the orbital radius R and the mechanical energy E of the planet.

2. Compute the quadrupole moment I_{ij} of the system.

3. What is the time averaged power radiated in gravitational waves?

4. The emission of gravitational waves gradually changes the orbital period of the planet. Determine the time evolution of the period T.

5. Estimate how long it takes for this process to change the year on Earth (orbital period around the Sun) by one second.

6. In reality, the star also moves when the planet orbits around it. Estimate the extra power radiated in gravitational waves due to the motion of the star.

   $G = 6.67 \times 10^{-11}\, m^3 kg^{-1} s^{-2}$
   $c = 3 \times 10^8\, m/s$
   $\hbar = 1.05 \times 10^{-34}\, m^2 kg/s$
   $M_{\text{Sun}} = 1.99 \times 10^{30}\, kg$
   $m_{\text{Earth}} = 5.97 \times 10^{24}\, kg$

3 Black hole shadow

Consider the Schwarzschild metric of a spherically symmetric black hole,

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_S}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{2}$$

1. Show that a photon moving in the equatorial plane $\theta = \frac{\pi}{2}$ follows a trajectory $r(\phi)$ obeying

   $$\left(\frac{dr}{d\phi}\right)^2 + V(r) = 0\,, \tag{3}$$

   and determine $V(r)$.

2. Show that $V(r)$ can be expressed solely in terms of r, r_S and the impact parameter b.
   Hint: You may study (3) at large r to relate b to other constants of motion.

3. Photons sent from infinity towards the black hole may or may not fall through the horizon depending on the impact parameter b. Compute the critical impact parameter b_c such that all photons with $b < b_c$ fall through the horizon. [1]

4. What is the impact parameter b_1 such that the incoming photon goes around the black hole once and then continues exactly in the same direction?
   You may define b_1/r_S implicitly in a form that could easily be given to a computer (for example, as a solution of an equation, possibly involving integrals).

[1] This means that a distant observer, at distance r_{obs} from the black hole, will see a circular black shadow with angular radius b_c/r_{obs} on a background image (the image will also be distorted by gravitational lensing).