Relativity and Cosmology I

Exam Solutions - 16/01/2023

> Do not worry about your difficulties in Mathematics. I can assure you mine are still greater.
>
> Albert Einstein

1. Surface

(a) The length of a generic curve $\gamma : (r(\lambda), \phi(\lambda))$ is given by

$$\begin{aligned} L_\gamma = \int_\gamma ds &= \int d\lambda \sqrt{g_{rr}(\lambda)\left(\frac{dr}{d\lambda}\right)^2 + g_{\phi\phi}(\lambda)\left(\frac{d\phi}{d\lambda}\right)^2} \\ &= \int d\lambda \sqrt{\left(1+\frac{r^2(\lambda)}{a^2}\right)\left(\frac{dr}{d\lambda}\right)^2 + r^2(\lambda)\left(\frac{d\phi}{d\lambda}\right)^2}\,. \end{aligned} \tag{1}$$

In this case we are interested in the curve $r(\lambda) = R$ and $\phi(\lambda) = \lambda$.

$$L_C = \int_0^{2\pi} R d\phi = 2\pi R\,. \tag{2}$$

(b) To compute the area inside the curve, we use the fact that the correct invariant area element is given by $\epsilon = \sqrt{|g|}\mathrm{d}r \wedge \mathrm{d}\phi$. We get

$$A_C = \int \sqrt{|g|} dr d\phi = 2\pi \int_0^R \sqrt{1+\frac{r^2}{a^2}} r dr \tag{3}$$

we change variables to $\rho = \frac{r}{a}$ and we get

$$A_C = \pi a^2 \int_0^{\frac{R}{a}} 2\rho\sqrt{1+\rho^2} d\rho = \frac{2}{3}\pi a^2 (1+\rho^2)^{\frac{3}{2}}\Big|_0^{\frac{R}{a}} = \frac{2\pi a^2}{3}\left[\left(1+\frac{R^2}{a^2}\right)^{\frac{3}{2}} - 1\right]. \tag{4}$$

(c) The distance from $r = 0$ to the curve C is given by

$$L_R = \int_0^R \sqrt{g_{rr}} dr = \int_0^R \sqrt{1+\frac{r^2}{a^2}} dr\,. \tag{5}$$

Given the hint in the exercise, we realize that the change of variables that will make our life simpler is $\frac{r}{a} = \sinh(x)$. The integral becomes

$$L_R = a \int_0^{\sinh^{-1}\left(\frac{R}{a}\right)} \sqrt{1+\sinh^2 x} \cosh x \; dx = a \int_0^{\sinh^{-1}\left(\frac{R}{a}\right)} \cosh^2 x \; dx\,. \tag{6}$$

Using the hint, we obtain

$$\begin{aligned} L_R &= a\left(\frac{x}{2} + \frac{1}{4}\sinh(2x)\right)\Big|_0^{\sinh^{-1}\left(\frac{R}{a}\right)} \\ &= \frac{a}{2}\left(\sinh^{-1}\left(\frac{R}{a}\right) + \frac{R}{a}\cosh\left(\sinh^{-1}\left(\frac{R}{a}\right)\right)\right) \\ &= \frac{a}{2}\left(\sinh^{-1}\left(\frac{R}{a}\right) + \frac{R}{a}\sqrt{1+\frac{R^2}{a^2}}\right) \end{aligned} \tag{7}$$

where we used $\sinh(2x) = 2\cosh(x)\sinh(x)$.

(d) A paraboloid is defined through the equation

$$z = c(x^2 + y^2) = cr^2\,. \tag{8}$$

The metric in $\mathbb{R}^3$ is

$$\begin{aligned} ds^2 &= \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}z^2 \\ &= \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + (2cr\mathrm{d}r)^2 \\ &= \left(1 + 4c^2r^2\right)\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \end{aligned} \tag{9}$$

To match with the given metric, we need $c = \frac{1}{2a}$, so that the paraboloid is

$$z = \frac{x^2 + y^2}{2a}\,. \tag{10}$$

2. Black Hole in 2+1 dimensions

(a) As the Riemann tensor components are provided, the Ricci scalar may be computed by rising and lowering indices along with utilizing its symmetries. The metric is diagonal, so the inverse components are

$$g^{tt} = -\frac{1}{f(r)}, \qquad g^{rr} = \frac{1}{g(r)}, \qquad g^{\phi\phi} = \frac{1}{r^2}. \tag{11}$$

With these, the candidate may pursuit evaluation of Ricci tensor components.

$$\begin{aligned} R_{rr} &= R^{\phi}{}_{r\phi r} + R^{t}{}_{rtr} = \frac{g'}{2rg} + \frac{1}{4f^2g}\left(ff'g' + g(f'^2 - 2ff'')\right) \\ R_{tt} &= R^{\phi}{}_{t\phi t} + R^{r}{}_{trt} = R^{\phi}{}_{t\phi t} + g^{rr}R_{rtrt} = R^{\phi}{}_{t\phi t} + g^{rr}g_{tt}R^{t}{}_{rtr} = \\ &= \frac{f'}{2rg} - \frac{f}{g}\frac{1}{4f^2g}\left(ff'g' + g(f'^2 - 2ff'')\right) \\ &= \frac{f'}{2rg} - \frac{1}{4fg^2}\left(ff'g' + g(f'^2 - 2ff'')\right) \\ R_{\phi\phi} &= R^{t}{}_{\phi t\phi} + R^{r}{}_{\phi r\phi} = g^{tt}R_{t\phi t\phi} + g^{rr}R_{r\phi r\phi} = g_{\phi\phi}\left(g^{tt}R^{\phi}{}_{t\phi t} + g^{rr}R^{\phi}{}_{r\phi r}\right) = \\ &= r^2\left(-\frac{1}{f}\frac{f'}{2rg} + \frac{1}{g}\frac{g'}{2rg}\right) = \frac{r}{2g}\left(\frac{g'}{g} - \frac{f'}{f}\right) \end{aligned} \tag{12}$$

The off-diagonal components are 0, as there are no non-zero Riemann tensor components R^i_{jik} with $j \neq k$.

Computing the Ricci scalar:

$$
\begin{aligned}
R = g^{\phi\phi}R_{\phi\phi} + g^{tt}R_{tt} + g^{rr}R_{rr} = \\
&= \frac{1}{2rg}\left(\frac{g'}{g} - \frac{f'}{f}\right) + \\
&- \frac{1}{f}\left(\frac{f'}{2rg} - \frac{1}{4fg^2}\left(ff'g' + g(f'^2 - 2ff'')\right)\right) + \\
&+ \frac{1}{g}\left(\frac{g'}{2rg} + \frac{1}{4f^2g}\left(ff'g' + g(f'^2 - 2ff'')\right)\right) = \\
&= \frac{1}{rg}\left(\frac{g'}{g} - \frac{f'}{f}\right) + \frac{1}{2f^2g^2}\left(ff'g' + g(f'^2 - 2ff'')\right)
\end{aligned}
\tag{13}
$$

So, finally, the tensor components and scalar are

$$
\begin{aligned}
R_{tt} &= \frac{f'}{2rg} - \frac{1}{4fg^2}\left(ff'g' + g(f'^2 - 2ff'')\right) \\
R_{rr} &= \frac{g'}{2rg} + \frac{1}{4f^2g}\left(ff'g' + g(f'^2 - 2ff'')\right) \\
R_{\phi\phi} &= \frac{r}{2g}\left(\frac{g'}{g} - \frac{f'}{f}\right) \\
R &= \frac{1}{rg}\left(\frac{g'}{g} - \frac{f'}{f}\right) + \frac{1}{2f^2g^2}\left(ff'g' + g(f'^2 - 2ff'')\right)
\end{aligned}
\tag{14}
$$

(b) Let's start by finding the components in terms of provided solution. First, let's eliminate $g(r)$ and $g'(r) = -\frac{f'(r)}{f^2(r)}$

$$
\begin{aligned}
R_{tt} &= \frac{ff'}{2r} - \frac{f}{4}\left(-\frac{(f')^2}{f} + \frac{1}{f}(f'^2 - 2ff'')\right) = \frac{ff'}{2r} + \frac{f''f}{2} \\
R_{rr} &= -\frac{f'}{2rf} + \frac{1}{4f}\left(-\frac{(f')^2}{f} + \frac{1}{f}(f'^2 - 2ff'')\right) = -\frac{f'}{2rf} - \frac{f''}{2f} \\
R_{\phi\phi} &= \frac{rf}{2}\left(-\frac{f'}{f} - \frac{f'}{f}\right) = -rf' \\
R &= -\frac{1}{f}R_{tt} + fR_{rr} + \frac{1}{r^2}R_{\phi\phi} = -\frac{2f'}{r} - f''
\end{aligned}
\tag{15}
$$

Plugging in $f' = \frac{2r}{l^2}, f'' = \frac{2}{l^2}$ one gets

$$
\begin{aligned}
R_{tt} &= \frac{ff'}{2r} + \frac{f''f}{2} = \frac{2f}{l^2} = 2\frac{r^2 - r_h^2}{l^4} \\
R_{rr} &= -\frac{f'}{2rf} - \frac{f''}{2f} = -\frac{2}{fl^2} = -2\frac{1}{r^2 - r_h^2} \\
R_{\phi\phi} &= -\frac{2r^2}{l^2} \\
R &= -\frac{2f'}{r} - f'' = -\frac{6}{l^2}
\end{aligned}
\tag{16}
$$

Careful observer shall notice that $R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu} = \frac{1}{3} R g_{\mu\nu}$. Plugging the result into Einstein equation

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= 8\pi G T_{\mu\nu} \\ -\frac{2}{l^2} g_{\mu\nu} + \frac{3}{l^2} g_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ \frac{1}{l^2} g_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ T_{\mu\nu} &= \frac{1}{8\pi G l^2} g_{\mu\nu} \end{aligned} \tag{17}$$

The energy-momentum tensor proportional to the metric may come from non-zero cosmological constant ($\Lambda = -\frac{1}{l^2}$) or vacuum energy or inflationary potential (see first question of the practice exam).

(c) Let's start with time-like geodesics. As in lectures, this may be done by computing $ds^2(\partial_\tau, \partial_\tau) = -1$ and using constant of motion.

$$-1 = -f(r) \left(\frac{dt}{d\tau} \right)^2 + g(r) \left(\frac{dr}{d\tau} \right)^2 + r^2 \left(\frac{d\phi}{d\tau} \right)^2 \tag{18}$$

Expanding $f(r), g(r)$ and marking $\frac{d}{d\tau}$ by dots:

$$-1 = -\frac{r^2 - r_h^2}{l^2} \dot{t}^2 + \frac{l^2}{r^2 - r_h^2} \dot{r}^2 + r^2 \dot{\phi}^2 \tag{19}$$

As t, ϕ are cyclic coordinates, one can associate constants of motion to them

$$\begin{aligned} E &= -g_{tt} \dot{t} = \frac{r^2 - r_h^2}{l^2} \dot{t} \\ L &= g_{\phi\phi} \dot{\phi} = r^2 \dot{\phi} \end{aligned} \tag{20}$$

which give the equation of motion

$$-1 = -\frac{l^2}{r^2 - r_h^2} E^2 + \frac{l^2}{r^2 - r_h^2} \dot{r}^2 + \frac{L^2}{r^2} \tag{21}$$

Reshuffling the terms give

$$\frac{\dot{r}^2}{2} + \frac{r^2 - r_h^2}{2l^2} \left(1 + \frac{L^2}{r^2} \right) = \frac{E^2}{2} \tag{22}$$

The derivation for null geodesics goes similarly, by starting with $ds^2(\partial_\lambda, \partial_\lambda) = 0$ and gives the equation

$$\frac{\dot{r}^2}{2} + \frac{r^2 - r_h^2}{2l^2} \frac{L^2}{r^2} = \frac{E^2}{2} \tag{23}$$

The potentials may be read of these equations:

$$\text{Time-like: } V(r) = \frac{r^2 - r_h^2}{2l^2} \left(1 + \frac{L^2}{r^2} \right)$$

$$\text{Light-like: } V(r) = \frac{r^2 - r_h^2}{2l^2} \frac{L^2}{r^2}$$

(d) For circular orbits one needs $\dot{r} = 0$ and $\ddot{r} = 0$. This corresponds to $V(r) = \frac{E^2}{2}$ and $V'(r) = 0$.

As E is (arbitrary) constant of motion, only the second condition is non-trivial. Evaluating $V'(r)$:

$$V'(r) = \begin{cases} \frac{r^4 + L^2 r_h^2}{l^2 r^3} & \text{for timelike geodesics} \\ \frac{L^2 r_h^2}{l^2 r^3} & \text{for null geodesics} \end{cases} \tag{24}$$

Both of these derivatives are non-negative for $r > 0$. Therefore, there are **no** circular orbits in this space.

(e) Unaccelerated observers move along timelike geodesics. These are described by (22) which may be transformed to

$$\frac{dr}{d\tau} = \pm\sqrt{E^2 - 2V(r)} \tag{25}$$

Firstly, notice that the timelike geodesic with maximal proper time (outside the horizon) starts at the horizon and ends at the horizon because there is no timelike geodesic that can stay outside the horizon forever due to $V'(r) > 0$. Therefore,

$$\tau_{outside}(E, L) = 2\int_{r_h}^{r_{max}} \frac{dr}{\sqrt{E^2 - 2V(r)}} \tag{26}$$

where r_{max} is the turning point determined by $E^2 = 2V(r_{max})$.

Secondly, let us argue that $\tau_{outside}(E, L)$ for fixed E is maximal for $L = 0$. Given an initial velocity $\dot{r}\Big|_{r=r_h} = \sqrt{E}$, the steeper the potential, the quicker it reaches the turning point because $\ddot{r} = -V'(r)$. In the present case, the potential $V(r) = \frac{r^2 - r_h^2}{2l^2}\left(1 + \frac{L^2}{r^2}\right)$ and its derivative (24) is growing strictly monotonically with L. Therefore, the maximal time trajectory corresponds to $L = 0$. [1]

This leaves the potential

$$V(r) = \frac{r^2 - r_h^2}{2l^2} \tag{27}$$

This is quadratic, with solution being simple harmonic motion, $r = A\sin(\omega\tau)$ where $A = \sqrt{\frac{E^2}{2} + \frac{r_h^2}{l^2}}$, and $\omega = \frac{1}{l}$. If the observer is initially at point $r = r_h$, the initial time is $\tau_0 = \frac{1}{\omega}\arcsin(\frac{r_h}{A})$, and the time at which it reaches turning point is $\tau_1 = \frac{\pi}{2\omega}$. Therefore, the total proper time spent on such motion outside the horizon is $\tau_{outside} = 2(\tau_1 - \tau_0)$. Finally τ_0 decreases with increasing E, with limit of 0 as $E \to \infty$, giving the maximal answer $\tau_{outside} = 2(\frac{\pi l}{2} - 0) = \pi l$.

To summarize, the timelike geodesic of maximal proper time outside the horizon describes the motion of an observer that starts at the horizon with a very large outgoing radial velocity, then the observer moves away from the horizon (to a very large distance) until it stops and falls back in after proper time

$$\tau_{outside} = \pi l \,.$$

[1] This can be argued more carefully as follows. Denote the velocity by $v(r) = \sqrt{E^2 - 2V(r)}$. Then, $\tau_{outside} = 2\int_{r_h}^{r_{max}} \frac{dr}{v(r)}$. Now consider the difference $\tau_{outside}(L_1) - \tau_{outside}(L_2) = 2\int_{r_h}^{r_1} \frac{dr}{v_1(r)} - 2\int_{r_h}^{r_2} \frac{dr}{v_2(r)}$ for fixed E. Since the potential increases with L, the turning points obey $r_1 > r_2$ for $L_1 < L_2$. Then, using $\delta r \equiv r_1 - r_2 > 0$, we can write $\tau_{outside}(L_1) - \tau_{outside}(L_2) = 2\int_{r_h}^{r_h+\delta r} \frac{dr}{v_1(r)} + 2\int_{r_h+\delta r}^{r_1} dr \left(\frac{1}{v_1(r)} - \frac{1}{v_2(r-\delta r)}\right)$. Both terms are positive. The first is obvious and the second follows from $V_2(r - \delta r) < V_1(r)$.

3. Explosion

(a) To compute the quadrupole moment, we start from the energy momentum tensor.

$$\begin{aligned} \text{before explosion:} \qquad & T^{00}(t, \vec{x}) = M\delta^{(3)}(\vec{x} - \vec{x}_0)\,, \\ \text{after explosion:} \qquad & T^{00}(t, \vec{x}) = \sum_{a=1}^{n} m_a \delta^{(3)}(\vec{x} - \vec{x}_0 - \vec{v}_a t)\,, \end{aligned} \tag{28}$$

where $M \equiv \sum_{a=1}^{n} m_a$ and $\vec{x}_0$ is the initial position of the isolated body. The quadrupole moment is instead given by

$$\begin{aligned} \text{before explosion:} \qquad & I_{ij}(t) = M x_0^i x_0^j \\ \text{after explosion:} \qquad & I_{ij}(t) = \sum_{a=1}^{n} m_a (x_0 + v_a t)^i (x_0 + v_a t)^j \end{aligned} \tag{29}$$

(b) We compute the second derivative of the quadrupole moment after the explosion

$$\ddot{I}_{ij}(t) = 2\sum_{a=1}^{n} m_a v_a^i v_a^j\,. \tag{30}$$

It clearly does not depend on the initial position of the body. Notice that this is true even during the explosion. To see that, we can write

$$I_{ij}(t) = \sum_{a=1}^{n} m_a x_a^i(t) x_a^j(t) \tag{31}$$

which leads to

$$\ddot{I}_{ij}(t) = 2\sum_{a=1}^{n} m_a \dot{x}_a^i(t)\dot{x}_a^j(t) + \sum_{a=1}^{n} m_a (\ddot{x}_a^i(t) x_a^j(t) + \ddot{x}_a^j(t) x_a^i(t)) \tag{32}$$

Now, shifting by a translation $x_a(t) \to x_a(t) + x_0$, we have

$$\ddot{I}_{ij}(t) \to \ddot{I}_{ij}(t) + x_0^i F^j(t) + x_0^j F^i(t) = \ddot{I}_{ij}(t) \tag{33}$$

because $F(t) = \sum_{a=1}^{n} m_a \ddot{x}_a(t) = 0$ since there is no external force applied to the system.

(c)

$$\begin{aligned} \ddot{I}^2 &= \ddot{I}_{ij}\ddot{I}_{kl}\delta^{ik}\delta^{jl} \\ &= 4\sum_{a,b=1}^{n} m_a m_b v_a^i v_a^j v_b^k v_b^l \delta_{ik}\delta_{jl} \\ &= 4\sum_{a,b=1}^{n} m_a m_b |\vec{v}_a \cdot \vec{v}_b|^2 \\ &= 4\sum_{a,b=1}^{n} m_a m_b |\vec{v}_a|^2 |\vec{v}_b|^2 \cos^2\theta_{ab} \\ &\leq 4\sum_{a,b=1}^{n} m_a m_b |\vec{v}_a|^2 |\vec{v}_b|^2 = 16E^2\,. \end{aligned} \tag{34}$$

The upper bound is saturated when $\theta_{ab} = 0, \pi$, which means pieces of the bomb are ejected along two opposite directions.

(d) To make this estimate, start from the expression for the power emitted in gravitational waves

$$P = -\frac{G}{5}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}\right\rangle \sim -G\frac{\ddot{I}^2}{T^2} \sim -G\frac{E^2}{T^2}\,. \tag{35}$$

The energy emitted in gravitational waves is thus

$$E_{\mathrm{GW}} = \frac{G}{c^5}\frac{E^2}{T} \approx 10^{-13} J\,. \tag{36}$$

In the last expression, it was important to reintroduce the speed of light c to restore standard units.