QUANTUM PHYSICS III

Problem Set 9 12 November 2024

1. Optical theorem in the Born approximation

Let $A^{(n)}$ denote the n'th term in the expansion of A with respect to the small parameter λ. The important observation is that the total cross section computed in the first Born approximation is actually of the second order in λ,

$$\int d\Omega |f^{(1)}|^2 = \sigma^{(2)} \, . \tag{1}$$

To check the validity of the optical theorem to the leading order, one should show that

$$\sigma^{(2)} = \frac{4\pi}{p} \text{Im } f^{(2)}(\mathbf{p} \to \mathbf{p}) \, . \tag{2}$$

The second-order term of the scattering amplitude is written as

$$f^{(2)}(\mathbf{p} \to \mathbf{p}') = -(2\pi)^2 m \int d^3\mathbf{p}'' \frac{\langle \mathbf{p}'|\hat{V}|\mathbf{p}''\rangle\langle \mathbf{p}''|\hat{V}|\mathbf{p}\rangle}{\frac{E_{p'}+E_p}{2} + i\epsilon - E_{p''}} \, . \tag{3}$$

Denote by $\tilde{V}(\mathbf{p}-\mathbf{q})$ the matrix element $\langle \mathbf{p}|\hat{V}|\mathbf{q}\rangle$. Taking the imaginary part of the expression (3) in the limit of forward scattering $\mathbf{p} = \mathbf{p}'$ gives

$$\begin{aligned} \text{Im } f^{(2)}(\mathbf{p} \to \mathbf{p}) &= -(2\pi)^2 m \int d^3\mathbf{p}'' |\tilde{V}(\mathbf{p}'' - \mathbf{p})|^2 \text{Im } \frac{1}{E_p + i\epsilon - E_{p''}} \\ &= 4\pi^3 m \int d^3\mathbf{p}'' |\tilde{V}(\mathbf{p}'' - \mathbf{p})|^2 \delta(E_p - E_{p''}) \, , \end{aligned} \tag{4}$$

where we made use of the relation

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \, . \tag{5}$$

On the other hand,

$$\begin{aligned} \sigma^{(2)} &= \int d\Omega |f^{(1)}|^2 = (2\pi)^4 m^2 \int d\Omega |\tilde{V}(\mathbf{p}' - \mathbf{p})|^2 \\ &= \frac{(2\pi)^4 m^2}{p^2} \int d^3\mathbf{p}' \delta(p' - p) |\tilde{V}(\mathbf{p}' - \mathbf{p})|^2 \\ &= \frac{(2\pi)^4 m}{p} \int d^3\mathbf{p}' \delta(E_p - E_{p'}) |\tilde{V}(\mathbf{p}' - \mathbf{p})|^2 \, , \end{aligned} \tag{6}$$

where we used the fact that

$$\int d\Omega = \frac{1}{p^2} \int d^3\mathbf{p}' \delta(p' - p) \, . \tag{7}$$

The direct inspection of the r.h.s. of eqs. (4) and (6) then gives eq. (2).

2. Scattering amplitude in a spherically-symmetric potential

1. In the first Born approximation the scattering amplitude is written as

$$f(\mathbf{p} \to \mathbf{p}') = -\frac{m}{2\pi} \int d^3\mathbf{x} V(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}} \,, \tag{8}$$

where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ is the momentum transfer. For a spherically symmetric potential this expression reduces to

$$\begin{aligned} f(\mathbf{p} \to \mathbf{p}') &= -m \int_0^\infty dr\ r^2 V(r) \int_0^\pi d\theta\ \sin\theta e^{-iqr\cos\theta} \\ &= m \int_0^\infty dr\ r^2 V(r) \frac{e^{iqr} - e^{-iqr}}{-iqr} \\ &= -\frac{2m}{q} \int_0^\infty dr\ r \sin(qr) V(r) \,. \end{aligned} \tag{9}$$

2. With the potential

$$V(r) = V_0 e^{-r^2/a^2} \tag{10}$$

the formula (9) becomes

$$\begin{aligned} f(\mathbf{p} \to \mathbf{p}') &= -\frac{2mV_0}{q} \int_0^\infty dr\ r e^{-r^2/a^2} \sin(qr) = -\frac{mV_0}{q} \int_{-\infty}^\infty dr\ r e^{-r^2/a^2} \sin(qr) \\ &= -\frac{mV_0 a^2}{2} \int_{-\infty}^\infty dr\ e^{-r^2/a^2} \cos(qr) = -\frac{mV_0 a^3}{2} \int_{-\infty}^\infty dr\ e^{-r^2} \cos(qar) \\ &= -\frac{mV_0 a^3}{4} \int_{-\infty}^\infty dr\ e^{-q^2a^2/4} \left(e^{-(r - iqa/2)^2} + e^{-(r + iqa/2)^2} \right) \\ &= -\frac{mV_0 a^3}{2} \sqrt{\pi} e^{-q^2a^2/4} \,, \end{aligned} \tag{11}$$

where in going to the second line we integrated by parts. Hence,

$$\frac{d\sigma}{d\Omega} = \frac{\pi m^2 V_0^2 a^6}{4} e^{-q^2a^2/2} \,. \tag{12}$$

3. Scattering in a square-well potential

1. Substituting the potential

$$V(r) = \begin{cases} -V_0 \,, & r < R \,, \\ 0 \,, & r > R \end{cases} \tag{13}$$

into eq. (9), we have

$$f(\mathbf{p} \to \mathbf{p}') = \frac{2mV_0}{q} \int_0^R dr\ r \sin qr \,. \tag{14}$$

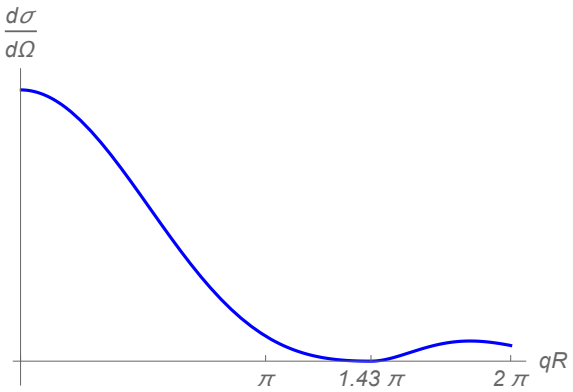

FIG. 1 – The plot of $d\sigma/d\Omega$ given by eq. (15) (not to scale).

Integrating by parts and taking a square, we obtain

$$\frac{d\sigma}{d\Omega} = 4R^6 m^2 V_0^2 \frac{(\sin qR - qR\cos qR)^2}{(qR)^6} \ . \tag{15}$$

The plot of this differential cross section in the units qR is shown in figure 1.

2. Note that the distribution (15) develops a zero at the value of qR such that $qR = \tan qR$, i.e., at $qR \approx 1.43\pi$. Hence, by measuring the angle θ_* in which no scattering occurs, one can extract the value of R,

$$R \approx \frac{1.43\pi}{2p \sin\frac{\theta_*}{2}} \ . \tag{16}$$

3. In order that R may be found from the measuring of the zero point of the differential cross section (15), the maximum value of qR, $2pR$, must be larger than 1.43π, or

$$\begin{aligned} E \geqslant \frac{\hbar^2}{2m_p}\left(\frac{1.43\pi}{2R}\right)^2 &= \frac{(1.43\pi)^2}{8}\frac{\hbar^2}{m_p c^2}\left(\frac{c}{R}\right)^2 \\ &= \frac{(1.43\pi)^2}{8}\cdot\frac{(6.58\cdot 10^{-22})^2}{938\cdot 10^6}\cdot\left(\frac{3\cdot 10^{10}}{5\cdot 10^{-13}}\right)^2 \\ &= 4.2 \text{ MeV} \ , \end{aligned} \tag{17}$$

where we restored $\hbar$ and c for numerical calculations.

4. From the formula for the momentum transfer, $q = 2p\sin\dfrac{\theta}{2}$, it follows that

$$d\Omega = d\phi \ d\cos\theta = 2\pi\frac{qdq}{p^2} \ . \tag{18}$$

Therefore,

$$\sigma = \int_0^{2p} \frac{d\sigma}{d\Omega}\frac{2\pi q dq}{p^2} \ . \tag{19}$$

Substituting eq. (15) we arrive after multiple integration by parts at

$$\sigma = \frac{2\pi}{p^2}(mV_0R^2)^2\left[1 - \frac{1}{(2pR)^2} + \frac{\sin 4pR}{(2pR)^3} - \frac{\sin^2 2pR}{(2pR)^4}\right] \ . \tag{20}$$

5. The slow scattering implies that the wave length $\lambda \sim p^{-1}$ of the scattered particles exceeds significantly the characteristic size of the potential. In our case this means $pR \ll 1$. Hence, to find the total cross section in this limit, we expand eq. (20) to the first nontrivial order in pR. This gives

$$\sigma = \frac{16\pi R^2}{9}(mV_0R^2)^2 \ . \tag{21}$$

   We observe that in the slow scattering regime the total cross section shows no dependence on the incident momentum of the particles. This is consistent with expectations, since the scattering amplitude (9) itself becomes independent of q in the limit $q \to 0$.

6. In the limit of fast scattering, $pR \gg 1$, we have

$$\sigma = \frac{2\pi}{p^2}(mV_0R^2)^2 \ . \tag{22}$$

   In agreement with expectations, the cross section goes to zero as the energy of the particles increases.

4. Cauchy's theorem and the completeness relation

Let us compute directly the integral of $\hat{G}(z)$ over the contour C. The integral over the semi-circle gives

$$\begin{aligned}\int_{circle} \frac{dz}{z-\hat{H}} &= \lim_{R\to\infty} R\int_0^{2\pi} \frac{d\phi i e^{i\phi}}{Re^{i\phi}-\hat{H}} \\ &= \lim_{R\to\infty}\left[\mathrm{Log}\left(e^{i\phi}-\frac{\hat{H}}{R}\right)_{\phi=2\pi} - \mathrm{Log}\left(e^{i\phi}-\frac{\hat{H}}{R}\right)_{\phi=0}\right] = 2\pi i \ .\end{aligned} \tag{23}$$

The integral over the branch cut is computed as follows,

$$\begin{aligned}\int_{branch\ cut} \hat{G}(z)dz &= \int_0^\infty dx \int dp|p\rangle\langle p|\left(\frac{1}{x+i\epsilon-E_p} - \frac{1}{x-i\epsilon-E_p}\right) \\ &= -2\pi i \int dp|p\rangle\langle p| \ ,\end{aligned} \tag{24}$$

where $|p\rangle$ denote eigenstates of the continuous spectrum of $\hat{H}$, and we used the completeness of the eigenstates. Summing up the contributions (23) and (24), we obtain

$$\begin{aligned}\oint_C \hat{G}(z)dz &= 2\pi i\left(\sum_n |n\rangle\langle n| + \int dp|p\rangle\langle p|\right) - 2\pi i\int dp|p\rangle\langle p| \\ &= 2\pi i\sum_n |n\rangle\langle n| \ .\end{aligned} \tag{25}$$

This result tells us that the integral is given by the sum of the residues computed at the poles of $\hat{G}(z)$ located inside the contour. Hence, we recovered the Cauchy theorem.

5*. The nucleus form factor

1. The potential $V(r)$ created by the charge distribution $\rho(r)$ satisfies Poisson's equation

$$\nabla^2 V(r) = \frac{1}{r}\frac{d^2}{dr^2}(rV(r)) = 4\pi e\rho(r)\ . \tag{26}$$

By eq. (9),

$$\begin{aligned} f(\mathbf{p}\to\mathbf{p}') &= -\frac{2m}{q}\int_0^\infty dr\ rV(r)\sin qr \\ &= \frac{2m}{q^2}\left[rV(r)\cos qr\right]_{r=0}^{r=\infty} - \frac{2m}{q^2}\int_0^\infty dr\ (rV(r))'\cos qr \\ &= \frac{2m}{q^2}\left[rV(r)\cos qr - \frac{1}{q}(rV(r))'\sin qr\right]_{r=0}^{r=\infty} + \frac{2m}{q^3}\int_0^\infty dr\ (rV(r))''\sin qr\ . \end{aligned} \tag{27}$$

It is clear that at large distances the real nucleus potential $V(r)$ becomes indistinguishable from the potential $V_0(r)$ created by the point nucleus. Therefore, there appears a problem of how to treat the scattering amplitudes computed for the Coulomb-like potential for which the standard scattering theory is inapplicable. In eq. (27), the problem is revealed by noticing that the boundary terms in the last line do not vanish. Instead of developing a new scattering theory, it takes much less efforts to *regularize* the potential, i.e., to assume that at very large distances $V(r)$ and $V_0(r)$ become falling off sufficiently fast to ensure the validness of the conventional scattering amplitudes. We do not discuss possible physical mechanisms of such suppression; in fact, we assume that it happens at the distances far beyond the scattering region we are interested in. For our results to make sense, one has to make sure that the physical observables are independent of a particular way of regularization (which is true), and that they are consistent with the results obtained within the rigorous approach (which is also true). Bearing the above in mind, we write

$$V(r)\ ,\ V_0(r) \sim \frac{Ze^2}{r}e^{-\alpha r}\ ,\quad r\to\infty\ . \tag{28}$$

The original potentials are restored in the limit $\alpha \to 0$. In this regularization

$$f(\mathbf{p}\to\mathbf{p}') = \frac{8\pi me}{q^3}\int_0^\infty dr\ r\rho(r)\sin qr\ , \tag{29}$$

while for the point-like nucleus

$$\begin{aligned} f_0(\mathbf{p}\to\mathbf{p}') &= \frac{2m}{q}\int_0^\infty dr\ Ze^2e^{-\alpha r}\sin qr \\ &= \frac{2mZe^2}{q}\frac{q}{q^2+\alpha^2}\ . \end{aligned} \tag{30}$$

We observe that after the scattering amplitude is computed, one can safely remove the regularization by sending α to 0. Then, comparing eqs. (29) and (30), we obtain

$$F(\mathbf{q}^2) = \frac{4\pi}{Ze}\int_0^\infty dr\ r^2\rho(r)\frac{\sin qr}{qr}\ . \tag{31}$$

The generalization to the case of non-spherically symmetric charge distributions is therefore

$$F(\mathbf{q}^2) = \frac{1}{Ze}\int d\mathbf{x}\,\rho(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}\ . \tag{32}$$

That is, the form factor is the Fourier transform of the charge distribution.

2. Differentiating eq. (31) with respect to q, we have

$$\frac{dF}{dq} = \frac{4\pi}{Ze}\int_0^\infty dr\ r^2\rho(r)\left[\frac{r\cos qr}{qr} - \frac{\sin qr}{q^2 r}\right]\ , \tag{33}$$

and

$$\frac{dF}{d(q^2)} = \frac{dF}{dq}\frac{dq}{d(q^2)} = \frac{1}{2q}\cdot\frac{dF}{dq}\ . \tag{34}$$

To find $dF/d(q^2)$ at $q^2 = 0$, we first compute

$$\begin{aligned}
&\lim_{q\to 0}\left[\frac{r\cos qr}{qr} - \frac{\sin qr}{q^2 r}\right] \\
&\quad = \lim_{q\to 0}\left[\frac{r\cdot\left(1-\frac{1}{2}(qr)^2\right)}{q^2 r} - \frac{qr - \frac{1}{6}(qr)^3}{q^3 r}\right] \\
&\quad = \lim_{q\to 0}\left(-\frac{r^2}{3}\right) = -\frac{r^2}{3}\ .
\end{aligned} \tag{35}$$

Then

$$\left.\frac{dF}{d(q^2)}\right|_{q^2=0} = -\frac{1}{6}\cdot\frac{1}{Ze}\int_0^\infty dr\ r^2\rho(r)\cdot 4\pi r^2 = -\frac{1}{6}\langle r^2\rangle\ . \tag{36}$$

Thus, the mean-square radius of the proton is found from the experimental data as

$$\langle r^2\rangle = -6\left.\frac{dF}{d(q^2)}\right|_{q^2=0}\ . \tag{37}$$

Numerically

$$\sqrt{\langle r^2\rangle} \approx 0.87\cdot 10^{-13}\ cm\ . \tag{38}$$

This quantity is also called the charge radius of the proton.