# QUANTUM PHYSICS III

**Solutions to Problem Set 8** 5 November 2024

## 1. On integrals involving the delta-function

1. Consider the integral

$$I = \int_{-\infty}^{\infty} dx\, f(x)\delta(ax^2 + bx + c)\ . \tag{1}$$

Denote the argument of the delta-function by $g(x)$. There are several possibilities :

— If the equation $g(x) = 0$ has no real roots, then the argument of the delta-function is never zero, hence $I = 0$.

— Suppose that the equation $g(x) = 0$ has two different real roots $x_{1,2}$. Near each of them the function $g(x)$ can be written as $g(x) = g'(x_{1,2})(x - x_{1,2}) + O((x - x_{1,2})^2)$. Let $O_1$ and $O_2$ be small neighborhoods of the points $x_1$ and $x_2$ correspondingly. The integral $I$ becomes

$$I = \int_{O_1} \frac{dy}{|g'(x_1 + y)|} f(x_1 + y)\delta(y) + \int_{O_2} \frac{dy}{|g'(x_2 + y)|} f(x_2 + y)\delta(y)\,, \tag{2}$$

where we made the change of variable $y = x - x_1$ in the first integral, $y = x - x_2$ in the second integral, and used the property of the delta-function

$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)\,, \tag{3}$$

with $\alpha$ some constant. Taking the integrals, we have

$$I = \frac{f(x_1)}{|g'(x_1)|} + \frac{f(x_2)}{|g'(x_2)|}\ . \tag{4}$$

Finally, $|g'(x_1)| = |g'(x_2)| = |a(x_1 - x_2)| = \sqrt{b^2 - 4ac}$, and

$$I = (f(x_1) + f(x_2))(b^2 - 4ac)^{-1/2}\ . \tag{5}$$

— Suppose now that $x_1 = x_2 = x_0$. Expanding $g(x)$ around $x_0$ and changing the variable $y = x - x_0$, we arrive at

$$I = \int_{-\infty}^{\infty} \frac{dy f(x_0 + y)}{|g'(x_0 + y)|}\delta(y) = \lim_{x \to x_0} \frac{f(x)}{2|a(x - x_0)|} = \begin{cases} \infty\,, & f(x_0) \neq 0\,, \\ \dfrac{f'(x_0)}{2|a|}\,, & f(x_0) = 0\,. \end{cases} \tag{6}$$

2. Recall that
$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \,, \tag{7}$$
where $i$ numerates the roots of the function $f$. In our case $E_p = \frac{p^2}{2m}$, and
$$\begin{aligned} \int d^3\mathbf{p}\; \delta(E_{p'} - E_p) f(\mathbf{p}) &= \int d\Omega dp p^2 \delta(E_p - E_{p'}) f(\mathbf{p}) \\ &= \int d\Omega dp p^2 \delta(p - p') \frac{2m}{2p'} f(\mathbf{p}) \\ &= mp' \int d\Omega f(\mathbf{n}) \,, \end{aligned} \tag{8}$$
where
$$\mathbf{n} = |\mathbf{p}'| \frac{\mathbf{p}}{|\mathbf{p}|} \tag{9}$$
is a vector of modulus $|\mathbf{p}'|$ in the direction of $\mathbf{p}$.

## 2. Free particle's Green function in three dimensions

1. By definition,
$$\hat{G}_0(z) = \frac{1}{z - \hat{H}_0} \,. \tag{10}$$
This means that
$$\hat{G}_0(z)|\mathbf{p}\rangle = \frac{1}{z - E_p}|\mathbf{p}\rangle \,, \quad E_p = \frac{p^2}{2m} \,. \tag{11}$$
Therefore,
$$\langle \mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle = \int d^3\mathbf{p} \langle \mathbf{x}|\hat{G}_0(z)|\mathbf{p}\rangle\langle \mathbf{p}|\mathbf{x}'\rangle = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{z - E_p}. \tag{12}$$

2. Let us first compute the angular part of the integral :
$$\begin{aligned} \langle \mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle &= 2\pi \frac{1}{(2\pi)^3} \int_0^\infty dp p^2 \int_0^\pi d\theta \sin\theta \frac{e^{ip|\mathbf{x}-\mathbf{x}'|\cos\theta}}{z - E_p} \\ &= -\frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2}{ip|\mathbf{x}-\mathbf{x}'|} \frac{1}{z - E_p} \left(e^{-ip|\mathbf{x}-\mathbf{x}'|} - e^{ip|\mathbf{x}-\mathbf{x}'|}\right) \\ &= -\frac{im}{2\pi^2|\mathbf{x}-\mathbf{x}'|} \int_{-\infty}^{\infty} dp p \frac{e^{ip|\mathbf{x}-\mathbf{x}'|}}{2mz - p^2} \,. \end{aligned} \tag{13}$$
The resulting integral can be computed by the method of residues. To this end, we close the contour of integration in the plane of complex $p$ as shown in figure 1. This does not change the value of the integral, since in the upper half-plane the integrand approaches zero exponentially fast when the radius of the semi-circle goes to infinity. The integrand has two poles at $p = \pm\sqrt{2mz}$. Recall that $z = E + i\epsilon$, $\epsilon > 0$, hence the pole contributing to the integral is the one at $p = +\sqrt{2mz}$. Thus,

$$
\begin{aligned}
\langle \mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle &= -\frac{im}{2\pi^2|\mathbf{x}-\mathbf{x}'|}2\pi i\ \mathrm{res}_{p=\sqrt{2mz}}\frac{pe^{ip|\mathbf{x}-\mathbf{x}'|}}{(p-\sqrt{2mz})(p+\sqrt{2mz})} \\
&= \frac{m}{2\pi}\frac{e^{i\sqrt{2mz}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\ .
\end{aligned} \tag{14}
$$

3. The formula (12) tells us that the Fourier transform of the function $G_0(z,\mathbf{x},\mathbf{x}') = \langle \mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle$ is

$$
G_0(z,\mathbf{p},\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')\frac{1}{z-E_p}\ . \tag{15}
$$

This implies in particular the conservation of the free particle momentum. We now use the momentum representation of the Green function to yield

$$
\begin{aligned}
&\langle \mathbf{x}|(z-\hat{H}_0)\hat{G}_0(z)|\mathbf{x}'\rangle = \int d^3\mathbf{p}d^3\mathbf{p}'d^3\mathbf{p}''\langle \mathbf{x}|\mathbf{p}\rangle\langle \mathbf{p}|z-\hat{H}_0|\mathbf{p}''\rangle\langle \mathbf{p}''|\hat{G}_0(z)|\mathbf{p}'\rangle\langle \mathbf{p}'|\mathbf{x}'\rangle \\
&= \frac{1}{(2\pi)^3}\int d^3\mathbf{p}d^3\mathbf{p}'d^3\mathbf{p}''e^{i\mathbf{p}\cdot\mathbf{x}}\delta(\mathbf{p}-\mathbf{p}'')(z-E_{p''})\delta(\mathbf{p}''-\mathbf{p}')\frac{1}{z-E_{p'}}e^{-i\mathbf{p}'\cdot\mathbf{x}'} \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p}d^3\mathbf{p}'e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{p}'\cdot\mathbf{x}'}\delta(\mathbf{p}-\mathbf{p}') \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p}e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')} = \delta(\mathbf{x}-\mathbf{x}')\ .
\end{aligned} \tag{16}
$$

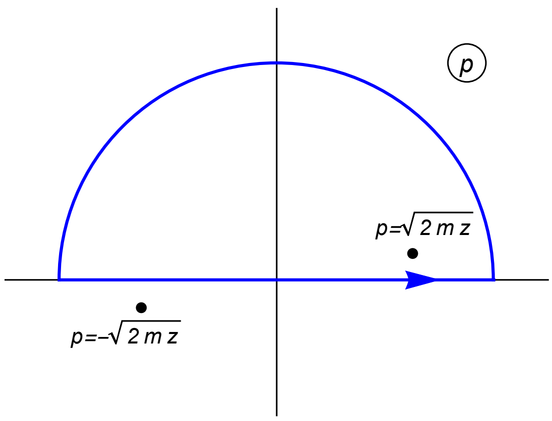


FIG. 1 – The contour of integration

4. The matrix element $G_0(z,\mathbf{x},\mathbf{x}')$ as a function of the complex variable $z$ has a branch cut along the real positive values of $z$. To calculate the difference between the points on the opposite sides of the branch cut, one should continue analytically the function $\sqrt{z}$ from the one side to another. This gives,

$$
\begin{aligned}
G_0(E+i\epsilon,\mathbf{x},\mathbf{x}') - G_0(E-i\epsilon,\mathbf{x},\mathbf{x}') &= -\frac{m}{2\pi|\mathbf{x}-\mathbf{x}'|}\left(e^{i\sqrt{2mE}\,|\mathbf{x}-\mathbf{x}'|} - e^{-i\sqrt{2mE}\,|\mathbf{x}-\mathbf{x}'|}\right) \\
&= -\frac{im}{\pi|\mathbf{x}-\mathbf{x}'|}\sin\left(\sqrt{2mE}\,|\mathbf{x}-\mathbf{x}'|\right)\ .
\end{aligned} \tag{17}
$$

5. For large values of $x = |\mathbf{x}|$, the modulus $|\mathbf{x} - \mathbf{x}'|$ can be expanded as

$$|\mathbf{x} - \mathbf{x}'| = x\sqrt{1 - \frac{2\mathbf{x}\cdot\mathbf{x}'}{x^2} + \frac{x'^2}{x^2}} \approx x - \frac{\mathbf{x}\cdot\mathbf{x}'}{x} \ . \tag{18}$$

Thus,

$$G_0(z, \mathbf{x}, \mathbf{x}') \approx -\frac{m}{2\pi}\frac{\exp\left[i\sqrt{2mz}\left(x - \frac{\mathbf{x}\cdot\mathbf{x}'}{x}\right)\right]}{x} \ , \quad x \to \infty \ . \tag{19}$$

# 3. Friedel sum rule

1. The eigenstates of the Hamiltonian $\hat{H}$ form an orthonormal basis of states. This fact allows us to write

$$\hat{G}(x + i\epsilon) = \frac{1}{x - \hat{H} + i\epsilon} = \sum_n \frac{|n\rangle\langle n|}{x - E_n + i\epsilon} \ . \tag{20}$$

Here by $|n\rangle$ we understand both the bound states and the scattering states. For the matrix element we have,

$$G_{nm}(x + i\epsilon) = \langle n|\hat{G}(x + i\epsilon)|m\rangle = \sum_n \frac{\delta_{nm}}{x - E_n + i\epsilon} \ , \tag{21}$$

or

$$G_{nn}(x + i\epsilon) = \frac{1}{x - E_n + i\epsilon} = \frac{d}{dx}\log G^{-1}_{nn}(x + i\epsilon) = -\frac{d}{dx}\log G_{nn}(x + i\epsilon) \ . \tag{22}$$

Now we turn to the function $N(x)$. Using the relation

$$\frac{1}{x + i\epsilon} = -i\pi\delta(x) + \mathcal{P}\frac{1}{x} \ , \tag{23}$$

we have,

$$N(x) = \sum_n \delta(x - E_n) = -\frac{1}{\pi}\sum_n \mathrm{Im}\frac{1}{x - E_n + i\epsilon} = -\frac{1}{\pi}\mathrm{Im}\sum_n G_{nn}(x + i\epsilon). \tag{24}$$

Substitution of the expression (22) then leads to

$$N(x) = \frac{1}{\pi}\mathrm{Im}\sum_n \frac{d}{dx}\log G_{nn}(x + i\epsilon) = \frac{1}{\pi}\frac{d}{dx}\mathrm{Im}\log\det\hat{G}(x + i\epsilon) \ . \tag{25}$$

2. From eq. (25) it follows that

$$\begin{aligned} N(x) - N_0(x) &= \frac{1}{\pi}\frac{d}{dx}\mathrm{Im}\left[\log\det\hat{G}(x + i\epsilon) - \log\det\hat{G}_0(x + i\epsilon)\right] \\ &= \frac{1}{\pi}\frac{d}{dx}\mathrm{Im}\log\det\hat{G}\hat{G}_0^{-1}(x + i\epsilon) \ . \end{aligned} \tag{26}$$

Writing $\det\hat{G}\hat{G}_0^{-1}$ as $|\det\hat{G}\hat{G}_0^{-1}|e^{i\arg\det\hat{G}\hat{G}_0^{-1}}$ gives

$$N(x) - N_0(x) = \frac{1}{\pi}\frac{d}{dx}\arg\det\hat{G}\hat{G}_0^{-1}(x + i\epsilon) \ . \tag{27}$$

## 4. Slow scattering in a gas

The problem concerns atom-atom scattering inside a gas. The condition

$$p \lesssim \frac{\hbar}{R} \tag{28}$$

implies $\mu v_r R \lesssim \hbar$, where $\mu = \frac{1}{2}m_p$ is the reduced mass of the two atoms, $\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity between the two atoms of velocities $\mathbf{v}_1$, $\mathbf{v}_2$, and $R = 4$ Å. In thermal equilibrium

$$\frac{1}{2}m_p\langle v^2\rangle = \frac{3}{2}kT \ , \tag{29}$$

with $k$ the Boltzmann constant and $T$ the temperature. The mean-square value of the relative speed $v_r$ is

$$\langle v_r^2\rangle = \langle(\mathbf{v}_1 - \mathbf{v}_2)^2\rangle = \langle v_1^2 + v_2^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2)\rangle = 2\langle v^2\rangle = \frac{6kT}{m_p} \ . \tag{30}$$

Thus,

$$\mu R v_r \approx \frac{m_p R}{2}\sqrt{\frac{6kT}{m_p}} \lesssim \hbar \ , \tag{31}$$

i.e.,

$$T \lesssim \frac{2\hbar^2}{3m_p c^2}\left(\frac{c}{R}\right)^2 \frac{1}{k} = 2\ K \ . \tag{32}$$