QUANTUM PHYSICS III

Solutions to Problem Set 2

17 September 2024

2.1 Classical limit of the harmonic oscillator

1. Recall that for the coherent states $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$. Hence,

$$
\begin{aligned}
\langle\hat{H}\rangle_\alpha &= \hbar\omega\langle\alpha|\hat{a}^\dagger\hat{a} + \frac{1}{2}|\alpha\rangle = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right) , \\
\langle\hat{H}^2\rangle_\alpha &= \hbar^2\omega^2\langle\alpha|\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)^2|\alpha\rangle = \hbar^2\omega^2\left(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4}\right) ,
\end{aligned}
\tag{1}
$$

and $\Delta\hat{H}_\alpha = \hbar\omega|\alpha|$. Classical approach is applicable when $\Delta\hat{H}_\alpha/\langle\hat{H}\rangle_\alpha \ll 1$, that is, when $|\alpha| \gg 1$.
For $\hat{N} = \hat{a}^\dagger\hat{a}$ we have similarly $\langle\hat{N}\rangle_\alpha = |\alpha|^2$, and

$$
\begin{aligned}
\langle\hat{N}^2\rangle_\alpha = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle &= |\alpha|^2\langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle \\
&= |\alpha|^2\langle\alpha|\hat{a}^\dagger\hat{a} + 1|\alpha\rangle = |\alpha|^4 + |\alpha|^2 .
\end{aligned}
\tag{2}
$$

Therefore, $\Delta\hat{N}_\alpha = |\alpha|$. Again, the classical limit $\Delta\hat{N}_\alpha/\langle\hat{N}\rangle_\alpha \ll 1$ is achieved if $|\alpha| \gg 1$.
For $\hat{X}$ and $\hat{P}$ we have,

$$
\begin{aligned}
\langle\hat{X}\rangle_\alpha &= \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|\hat{a} + \hat{a}^\dagger|\alpha\rangle = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\alpha , \\
\langle\hat{P}\rangle_\alpha &= -i\sqrt{\frac{m\omega\hbar}{2}}\langle\alpha|\hat{a} - \hat{a}^\dagger|\alpha\rangle = \sqrt{2m\hbar\omega}\,\mathrm{Im}\,\alpha ,
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
\langle\hat{X}^2\rangle_\alpha &= \frac{\hbar}{2m\omega}\langle\alpha|(\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger)|\alpha\rangle = \frac{\hbar}{2m\omega}(4\,(\mathrm{Re}\,\alpha)^2 + 1) , \\
\langle\hat{P}^2\rangle_\alpha &= -\frac{m\omega\hbar}{2}\langle\alpha|(\hat{a} - \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger)|\alpha\rangle = \frac{m\omega\hbar}{2}(4\,(\mathrm{Im}\,\alpha)^2 + 1) ,
\end{aligned}
\tag{4}
$$

from which one gets

$$
\Delta\hat{X}_\alpha = \sqrt{\frac{\hbar}{2m\omega}} , \quad \Delta\hat{P}_\alpha = \sqrt{\frac{m\omega\hbar}{2}} . \tag{5}
$$

Once more, the classical approach is valid when $\Delta\hat{X}_\alpha/\langle\hat{X}\rangle_\alpha \ll 1$, $\Delta\hat{P}_\alpha/\langle\hat{P}\rangle_\alpha \ll 1$, or, equivalently, $|\alpha| \gg 1$. Finally, we see that $\Delta\hat{X}_\alpha \cdot \Delta\hat{P}_\alpha = \hbar/2$, i.e., the coherent states minimize the uncertainty relation for the operators $\hat{X}$ and $\hat{P}$.

2. The frequency of the pendulum oscillations is given by $\omega = 2\pi/T$, where $T = 2\pi\sqrt{l/g}$ is the period of oscillation. Using the results of the p. 1, we have,

$$\langle\hat{X}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\alpha(t) = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\left(\alpha(0)e^{-i\omega t}\right)\ . \tag{6}$$

The amplitude of oscillations x_M is nothing but the maximal value of $\langle\hat{X}\rangle_\alpha(t)$, hence

$$|\alpha| = \sqrt{\frac{m\omega}{2\hbar}}x_M \approx 2.2\cdot 10^{15} \gg 1\ , \tag{7}$$

as expected for the classical pendulum. Similarly,

$$\Delta\hat{X}_\alpha = \sqrt{\frac{\hbar}{2m\omega}} \approx 2.2\cdot 10^{-18}\,m \ll x_M\ ,\quad \Delta\hat{P}_\alpha = \sqrt{\frac{m\hbar\omega}{2}} \approx 2.2\cdot 10^{-17}\,\frac{kg\cdot m}{s}\ , \tag{8}$$

and

$$\langle\hat{H}\rangle_\alpha \approx 4\cdot 10^{-3}\frac{kg\cdot m^2}{s^2}\ ,\quad \Delta\hat{H}_\alpha \approx 10^{-18}\frac{kg\cdot m^2}{s^2}\ , \tag{9}$$

and the relation $\Delta\hat{H}_\alpha/\langle\hat{H}\rangle_\alpha \ll 1$ holds.

2.2 Squeezed states

1. As any quantum state, the squeezed states $|\Psi_\lambda\rangle$ must be normalized to unity. For the wave function in the x-representation, $\Psi_\lambda(x) = C\Psi_\alpha(\lambda x)$, this gives

$$1 = \int_{-\infty}^{\infty}|\Psi_\lambda(x)|^2\,dx = C^2\int_{-\infty}^{\infty}|\Psi_\alpha(\lambda x)|^2\,dx = \frac{C^2}{\lambda}\ \Rightarrow\ C^2 = \lambda\ , \tag{10}$$

since the coherent state wave function $\Psi_\alpha(x)$ is in turn normalized.

2. Let us write $\langle\hat{X}\rangle$ in terms of the wave function $\Psi_\lambda(x)$:

$$\langle\hat{X}\rangle = \int_{-\infty}^{\infty}\langle\Psi_\lambda|\hat{X}|x\rangle\langle x|\Psi_\lambda\rangle dx = \int_{-\infty}^{\infty}x|\Psi_\lambda(x)|^2\,dx\ . \tag{11}$$

Now it is easy to express $\langle\hat{X}\rangle$ through $\langle\hat{X}\rangle_\alpha$,

$$\langle\hat{X}\rangle = C^2\int_{-\infty}^{\infty}x|\Psi_\alpha(\lambda x)|^2\,dx = \frac{C^2\langle X\rangle_\alpha}{\lambda^2} = \frac{\langle\hat{X}\rangle_\alpha}{\lambda}\ . \tag{12}$$

To calculate $\langle\hat{P}\rangle$, it is convenient to use the momentum basis for the state $|\Psi_\lambda\rangle$,

$$\langle\hat{P}\rangle = \int_{-\infty}^{\infty}\langle\Psi_\lambda|\hat{P}|p\rangle\langle p|\Psi_\lambda\rangle dp = \int_{-\infty}^{\infty}p|\Psi_\lambda(p)|^2\,dp\ . \tag{13}$$

Then, one can write $\Psi_\lambda(p)$ explicitly as a Fourier image of $\Psi_\lambda(x)$ and express the latter through $\Psi_\alpha(x)$:

$$\begin{aligned}\langle\hat{P}\rangle &= \int_{-\infty}^{\infty}p\left|\int_{-\infty}^{\infty}\Psi_\lambda(x)e^{ipx}dx\right|^2\,dp = \frac{C^2}{\lambda^2}\int_{-\infty}^{\infty}p\left|\Psi_\alpha\left(\frac{p}{\lambda}\right)\right|^2\,dp\\ &= C^2\langle\hat{P}\rangle_\alpha = \lambda\langle\hat{P}\rangle_\alpha\ .\end{aligned} \tag{14}$$

Similarly, we have

$$\langle \hat{X}^2 \rangle = \frac{\langle \hat{X}^2 \rangle_\alpha}{\lambda^2} \ , \quad \langle \hat{P}^2 \rangle = \lambda^2 \langle \hat{P}^2 \rangle_\alpha \ . \tag{15}$$

The dispersions are given by

$$\Delta\hat{X} = \frac{\Delta\hat{X}_\alpha}{\lambda}, \;\; \Delta\hat{P} = \lambda\,\Delta\hat{P}_\alpha \;\Rightarrow\; \Delta\hat{X}\cdot\Delta\hat{P} = \frac{\hbar}{2} \ . \tag{16}$$

The conclusion is that the squeezed state $|\Psi_\lambda\rangle$ minimizes the uncertainty relation. Note, however, that so far we proved this only at one (initial) moment of time. Our goal for the rest of the exercise is to find out the behavior of the dispersions (16) as the state $|\Psi_\lambda\rangle$ evolves in time.

3. In the coordinate representation the momentum operator $\hat{P}$ acts on a wave function $\Psi(x)$ by $-i\hbar\cdot\partial/\partial x$. The values $\Psi(x)$ and $\Psi(x{+}a)$ are related via the Taylor expansion,

$$\Psi(x+a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{\partial^n}{\partial x^n}\Psi(x) \ . \tag{17}$$

We can think of the r.h.s. of this equation as the result of applying some operator $\hat{T}(a)$ to $\Psi(x)$:

$$\Psi(x+a) = \hat{T}(a)\Psi(x) \ . \tag{18}$$

Comparing with eq. (17) it is easy to see that $\hat{T}(a)$ is given explicitly by

$$\hat{T}(a) = e^{\frac{i}{\hbar}a\cdot\hat{P}} \ , \tag{19}$$

and this is the familiar expression for the displacement operator.
Now we want to use this logic when looking for a *scaling* operator $\hat{S}_\lambda$ such that

$$\hat{S}_\lambda\Psi_\alpha(x) = \sqrt{\lambda}\Psi_\alpha(\lambda x) \ . \tag{20}$$

By analogy with eq. (19), we write

$$\hat{S}_\lambda = e^{\frac{i}{\hbar}f(\lambda)\cdot\hat{G}} \ , \tag{21}$$

where $f(\lambda)$ is some unknown function of λ, and $\hat{G}$ is an operator of infinitesimal scaling. Since $\hat{S}_\lambda$ is required to be unitary, $\hat{G}$ must be hermitian.
The key observation now is that any scaling of x can be regarded as a shift of $\log|x|$. Thinking of $\log|x|$ as a new variable to be shifted in the function $\Psi_\alpha(e^{\log|x|})$, one can guess $\hat{G}$ to be of the form (in the x-representation)

$$\hat{G} \sim -i\hbar\frac{\partial}{\partial\log|x|} = \hat{X}\hat{P} \ . \tag{22}$$

However, this operator is not hermitian, since $\hat{X}$ and $\hat{P}$ do not commute. To restore hermiticity, we can use instead the symmetric combination of $\hat{X}$ and $\hat{P}$:

$$\hat{G} = \frac{1}{2}\left(\hat{X}\hat{P} + \hat{P}\hat{X}\right) = \hat{X}\hat{P} - \frac{i\hbar}{2} = -i\hbar\left(x\frac{\partial}{\partial x} + \frac{1}{2}\right) \ . \tag{23}$$

Thus, the operator (21) with $\hat{G}$ given by eq. (23) is the desired unitary operator. It remains to find the function $f(\lambda)$. To this end, we compute explicitly the action of $\hat{S}_\lambda$ on $\Psi_\alpha(x)$,

$$\begin{aligned}\hat{S}_\lambda \Psi_\alpha(x) &= e^{\frac{f(\lambda)}{2}} e^{f(\lambda) x \frac{\partial}{\partial x}} \Psi_\alpha(x) = e^{\frac{f(\lambda)}{2}} \sum_{n=0}^{\infty} \frac{f(\lambda)^n}{n!} \left(x \frac{\partial}{\partial x} \right)^n \Psi_\alpha(x) \\ &= e^{\frac{f(\lambda)}{2}} \sum_{n=0}^{\infty} \frac{f(\lambda)^n}{n!} \frac{\partial^n}{\partial (\log |x|)^n} \Psi_\alpha(e^{\log |x|}) = e^{\frac{f(\lambda)}{2}} \Psi_\alpha(e^{\log |x| + f(\lambda)}) \\ &= e^{\frac{f(\lambda)}{2}} \Psi_\alpha(e^{f(\lambda)} x) \ .\end{aligned} \tag{24}$$

Comparing this with eq. (20), we find

$$f(\lambda) = \log \lambda \ . \tag{25}$$

4. The time-dependent vectors $|\Psi_\lambda(t)\rangle$ and $|\Psi_\alpha(t)\rangle$ are related by

$$|\Psi_\lambda(t)\rangle = \hat{U}_t |\Psi_\lambda\rangle = \hat{U}_t \hat{S}_\lambda |\Psi_\alpha\rangle = \hat{U}_t \hat{S}_\lambda \hat{U}_t^{-1} \hat{U}_t |\Psi_\alpha\rangle = \hat{S}_\lambda(-t) |\Psi_\alpha(t)\rangle \ , \tag{26}$$

with $\hat{S}_\lambda(-t)$ the scaling operator in the Heisenberg picture.

5. Substituting the expressions for $\hat{X}$ and $\hat{P}$ through the ladder operators,

$$\begin{aligned}\hat{X} &= \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^\dagger \right) \ , \\ \hat{P} &= -i \sqrt{\frac{\hbar m\omega}{2}} \left(\hat{a} - \hat{a}^\dagger \right) \ ,\end{aligned} \tag{27}$$

into eq. (23), we obtain,

$$\hat{G} = -i\hbar \frac{\hat{a}^2 - \hat{a}^{\dagger 2}}{2} \ . \tag{28}$$

Plugging this into eq. (21), we have,

$$\hat{S}_\lambda = \exp\left(\frac{\log \lambda}{2} \left(\hat{a}^2 - \hat{a}^{\dagger 2} \right) \right) \ . \tag{29}$$

Let us now evolve it in time,

$$\begin{aligned}\hat{S}_\lambda(-t) &= \hat{U}_t \hat{S}_\lambda \hat{U}_t^{-1} = \hat{U}_t e^{\frac{i}{\hbar} \log \lambda \cdot \hat{G}} \hat{U}_t^{-1} \\ &= \hat{U}_t \left(1 + \frac{i}{\hbar} \log \lambda \cdot \hat{G} + \frac{1}{2!} \left(\frac{i}{\hbar} \log \lambda \right)^2 \hat{G}\hat{G} + ... \right) \hat{U}_t^{-1} \\ &= 1 + \frac{i}{\hbar} \log \lambda \cdot \hat{U}_t \hat{G} \hat{U}_t^{-1} + \frac{1}{2!} \left(\frac{i}{\hbar} \log \lambda \right)^2 \left(\hat{U}_t \hat{G} \hat{U}_t^{-1} \right) \left(\hat{U}_t \hat{G} \hat{U}_t^{-1} \right) + ... \\ &= e^{\frac{i}{\hbar} \log \lambda \cdot \hat{U}_t \hat{G} \hat{U}_t^{-1}} \ .\end{aligned} \tag{30}$$

Hence, $\hat{S}_\lambda(-t)$ is obtained by simply converting the ladder operators $\hat{a}$, $\hat{a}^\dagger$ in eq. (29) to $\hat{a}(-t)$ and $\hat{a}^\dagger(-t)$,

$$\begin{aligned}\hat{S}_\lambda(-t) &= \exp\left(\frac{\log \lambda}{2} \left(\hat{a}^2(-t) - \hat{a}^{\dagger 2}(-t) \right) \right) \\ &= \exp\left(\frac{\log \lambda}{2} \left(e^{2i\omega t} \hat{a}^2 - e^{-2i\omega t} \hat{a}^{\dagger 2} \right) \right) \ .\end{aligned} \tag{31}$$

6. To compute the products $\hat{S}^{\dagger}_{\lambda}(-t)\hat{a}\hat{S}_{\lambda}(-t)$ and $\hat{S}^{\dagger}_{\lambda}(-t)\hat{a}^{\dagger}\hat{S}_{\lambda}(-t)$, we make use of the following identity, valid for some operators $\hat{O}$ and $\hat{b}$,

$$e^{-\hat{O}}\hat{b}e^{\hat{O}} = \hat{b} - \left[\hat{O},\hat{b}\right] + \frac{1}{2!}\left[\hat{O}\left[\hat{O},\hat{b}\right]\right] + \ldots + \frac{(-1)^n}{n!}\left[\hat{O}\left[\ldots\left[\hat{O},\hat{b}\right]\ldots\right]\right] + \ldots \,. \tag{32}$$

In our case

$$\hat{O} = \frac{1}{2}\log\lambda\left(e^{2i\omega t}\hat{a}^2 - e^{-2i\omega t}\hat{a}^{\dagger 2}\right) \;,\;\; \hat{b} = \hat{a} \text{ or } \hat{a}^{\dagger} \;. \tag{33}$$

We have,

$$\begin{aligned} [\hat{O},a] &= \log\lambda\, e^{-2i\omega t}a^{\dagger} \;, & [\hat{O},a^{\dagger}] &= \log\lambda\, e^{2i\omega t}a \;, \\ [\hat{O},[\hat{O},a]] &= (\log\lambda)^2 a \;, & [\hat{O},[\hat{O},a^{\dagger}]] &= (\log\lambda)^2 a^{\dagger} \;, \\ [\hat{O},[\hat{O},[\hat{O},a]]] &= (\log\lambda)^3 e^{-2i\omega t}a^{\dagger} \;, & [\hat{O},[\hat{O},[\hat{O},a^{\dagger}]]] &= (\log\lambda)^3 e^{2i\omega t}a \;, \\ &\ldots & &\ldots \end{aligned} \tag{34}$$

Summing this up, we arrive at

$$\begin{aligned} \hat{S}^{\dagger}_{\lambda}(-t)\hat{a}\hat{S}_{\lambda}(-t) &= -\,\hat{a}^{\dagger}e^{-2i\omega t}\sinh\log\lambda + \hat{a}\cosh\log\lambda \;, \\ \hat{S}^{\dagger}_{\lambda}(-t)\hat{a}^{\dagger}\hat{S}_{\lambda}(-t) &= -\,\hat{a}e^{2i\omega t}\sinh\log\lambda + \hat{a}^{\dagger}\cosh\log\lambda \;. \end{aligned} \tag{35}$$

7. Let us finally combine all our prerequisites. To compute the dispersions $\Delta\hat{X}$, $\Delta\hat{P}$, one needs to find the expressions for the following relations,

$$\begin{aligned} \langle\Psi_{\lambda}(t)|\hat{X}|\Psi_{\lambda}(t)\rangle &= \langle\Psi_{\alpha}(t)|\hat{S}^{\dagger}(-t)\hat{X}\hat{S}(-t)|\Psi_{\alpha}(t)\rangle \;, \\ \langle\Psi_{\lambda}(t)|\hat{P}|\Psi_{\lambda}(t)\rangle &= \langle\Psi_{\alpha}(t)|\hat{S}^{\dagger}(-t)\hat{P}\hat{S}(-t)|\Psi_{\alpha}(t)\rangle \;, \\ \langle\Psi_{\lambda}(t)|\hat{X}^2|\Psi_{\lambda}(t)\rangle &= \langle\Psi_{\alpha}(t)|\hat{S}^{\dagger}(-t)\hat{X}\hat{S}(-t)\hat{S}^{\dagger}(-t)\hat{X}\hat{S}(-t)|\Psi_{\alpha}(t)\rangle \;, \\ \langle\Psi_{\lambda}(t)|\hat{P}^2|\Psi_{\lambda}(t)\rangle &= \langle\Psi_{\alpha}(t)|\hat{S}^{\dagger}(-t)\hat{P}\hat{S}(-t)\hat{S}^{\dagger}(-t)\hat{P}\hat{S}(-t)|\Psi_{\alpha}(t)\rangle \;. \end{aligned} \tag{36}$$

They are computed easily by the means of eqs. (27), (35), and knowing that

$$\begin{aligned} \langle\Psi_{\alpha}(t)|\hat{a}|\Psi_{\alpha}(t)\rangle &= \alpha(t) = \alpha e^{-i\omega t} \;, \\ \langle\Psi_{\alpha}(t)|\hat{a}^{\dagger}|\Psi_{\alpha}(t)\rangle &= \alpha^{*}(t) = \alpha^{*}e^{i\omega t} \;. \end{aligned} \tag{37}$$

For example,

$$\begin{aligned} \langle\Psi_{\lambda}(t)|\hat{X}|\Psi_{\lambda}(t)\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left(2\mathrm{Re}\left(\alpha e^{-i\omega t}\right)\cosh\log\lambda - 2\mathrm{Re}\left(\alpha e^{i\omega t}\right)\sinh\log\lambda\right) \;, \\ \langle\Psi_{\lambda}(t)|\hat{P}|\Psi_{\lambda}(t)\rangle &= \sqrt{\frac{\hbar m\omega}{2}}\left(2\mathrm{Im}\left(\alpha e^{-i\omega t}\right)\cosh\log\lambda + 2\mathrm{Im}\left(\alpha e^{i\omega t}\right)\sinh\log\lambda\right) \;. \end{aligned} \tag{38}$$

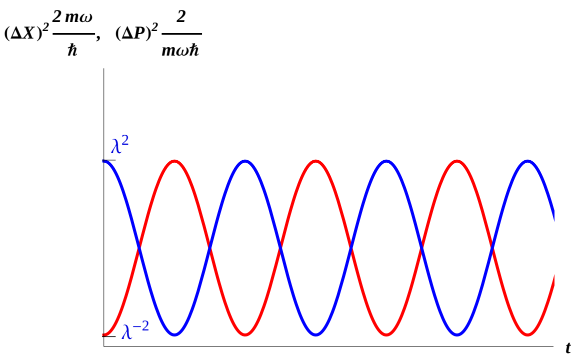
The answer is

$$\begin{aligned} (\Delta\hat{X})^2 &= \frac{\hbar}{2m\omega}\left[\cosh(2\log\lambda) - \cos(2\omega t)\sinh(2\log\lambda)\right] \;, \\ (\Delta\hat{P})^2 &= \frac{m\omega\hbar}{2}\left[\cosh(2\log\lambda) + \cos(2\omega t)\sinh(2\log\lambda)\right] \;. \end{aligned} \tag{39}$$

(Note the magical contraction of all α-dependent terms.) Thus,

$$\Delta\hat{X} \cdot \Delta\hat{P} = \frac{\hbar}{2}\sqrt{1 + \sinh^2(2\log\lambda)(1 - \cos^2(2\omega t))}\,. \tag{40}$$

First of all, we see that the uncertainty relation (40) is minimized at $t = 0$. This is in accordance with the result (16). At arbitrary $t > 0$ it is not minimized but neither does it grow uncontrollably. In fact, the quantity $\Delta\hat{X} \cdot \Delta\hat{P}$ oscillates around the average value $\hbar/2\sqrt{1 + 0.5\sinh^2(2\log\lambda)}$, and hits the minimum value $\hbar/2$ at any $t_n = \frac{\pi n}{2\omega}$ with n an integer. Thus, although the squeezed states do not minimize the uncertainty relation at any time, their dispersions are confined in a constant region determined by the parameter λ, and this property makes them close to the classical systems. The behavior of $(\Delta\hat{X})^2$ and $(\Delta\hat{P})^2$ is shown on the figure below.



8. Let us use the following dimensionless notation for clarity:

$$\begin{cases} \hat{x} = \sqrt{\dfrac{2m\omega}{\hbar}}\,\hat{X} = \hat{a} + \hat{a}^\dagger \\ \hat{p} = \sqrt{\dfrac{2}{\hbar m\omega}}\,\hat{P} = -i(\hat{a} - \hat{a}^\dagger)\,, \end{cases} \tag{41}$$

with the commutation relation $[\hat{x}, \hat{p}] = 2i$.

Consider the following change of variable:

$$\begin{pmatrix} \hat{x}' \\ \hat{p}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \tag{42}$$

It is a canonical transformation only if the commutation relation is preserved. It is trivial to see that

$$ad - bc = 1 \tag{43}$$

is a sufficient condition.

To satisfy this relation, one can look at solutions of the form $a = d = \cos\phi$, and $-b = c = \sin\phi$, with $\phi(t)$ a time dependent phase left to determine.

One can write the transformed operators as a function of the ladder operators:

$$\begin{cases} \hat{x}' = \hat{a}e^{i\phi} + \hat{a}^\dagger e^{-i\phi} \\ \hat{p}' = -i(\hat{a}e^{i\phi} - \hat{a}^\dagger e^{-i\phi}) \end{cases} \tag{44}$$

One could compute $\Delta x' \Delta p'$ explicitely, but the calculation is rather cumbersome. From the previous question, we saw that the uncertainty relation is minimized for $t = 0$, however not for finite t. A way to minimize it for all $t > 0$ would be to "cancel" the effect of time evolution with the canonical transformation. In the Heisenberg picture, the ladder operators just pick up a phase with time:

$$\begin{cases} \hat{a}(t) = \hat{a} e^{-i\omega t} \\ \hat{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega t} \end{cases} \tag{45}$$

From equation (44), the canonical transformation also only adds a phase shift ϕ to the ladder operators. We can use it to cancel the time evolution, with:

$$\phi(t) = \omega t, \tag{46}$$

which guarantees that the uncertainty relation will be minimized at all times $t > 0$.