QUANTUM PHYSICS III

Solutions to Problem Set 11 26 November 2024

1. Properties of spherical Bessel functions

1. We can use the Taylor series for $\sin\rho = \sum_{n=0}^{\infty} \frac{(-1)^n \rho^{2n+1}}{(2n+1)!}$, and the fact that :

$$\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\right)^l \rho^{2n} = \begin{cases} \frac{(2n)!!}{(2n-2l)!!}\rho^{2n-2l} & \text{for } n \geq l, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

which can easily be proved by induction. We thus have a series expansion for j_l :

$$j_l(\rho) = \sum_{n=l}^{\infty} \frac{(-1)^{n+l}(2n)!!}{(2n+1)!(2n-2l)!!}\rho^{2n-l}. \tag{2}$$

As $\rho \to 0$, we can keep only the leading order ($n = l$), hence :

$$j_l(\rho) \sim \frac{(2l)!!\rho^l}{(2n+1)!} = \frac{\rho^l}{(2l+1)!!}. \tag{3}$$

The case of y_l is similar, expanding the cosine in a Taylor series.

2. We can prove this by induction. The relation holds for $l = 0$:

$$j_0(\rho) = \frac{\sin\rho}{\rho}, \tag{4}$$

$$y_0(\rho) = -\frac{\cos\rho}{\rho}. \tag{5}$$

Assuming the relation holds for a given l, we have :

$$\begin{aligned} j_{l+1}(\rho) &= (-1)^{l+1}\rho^{l+1}\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\right)^{l+1}\frac{\sin\rho}{\rho} \\ &= (-1)^{l+1}\rho^{l+1}\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\right)(-1)^l\rho^{-l}j_l(\rho) \\ &\sim -\rho^l\frac{\mathrm{d}}{\mathrm{d}\rho}\rho^{-l-1}\sin(\rho - l\frac{\pi}{2}) \\ &= -(-l-1)\rho^{-2}\sin(\rho - l\frac{\pi}{2}) - \rho^{-1}\cos(\rho - l\frac{\pi}{2}) \\ &\sim \frac{1}{\rho}\sin(\rho - (l+1)\frac{\pi}{2}). \end{aligned}$$

In the last line, we neglected the first term, because $1/\rho^2$ decays faster than $1/\rho$ at large ρ. The asymptotic for y_l can be proved in a similar manner.

2. Scattering phase shift in a Yukawa potential

1. One has to start with the decomposition of the scattering amplitude in spherical harmonics :

$$f(\mathbf{p}' \leftarrow \mathbf{p}) = \sum_{l=0}^{\infty}(2l+1)f_l(p)P_l(\cos\theta). \tag{6}$$

With the orthogonality relation of the hint, it is trivial to see that

$$\begin{aligned} f_l &= \frac{1}{2}\int_0^{2\pi} \mathrm{d}\theta\ \sin\theta f(\mathbf{p}' \leftarrow \mathbf{p})P_l(\cos\theta) \\ &= -\frac{mV_0}{\mu}\int_0^{2\pi} \mathrm{d}\theta\ \frac{P_l(\cos\theta)\sin\theta}{2p^2(1-\cos\theta)+\mu^2} \\ &= -\frac{mV_0}{\mu}\int_{-1}^{1} \mathrm{d}\zeta\ \frac{P_l(\zeta)}{2p^2(1-\zeta)+\mu^2} \\ &= -\frac{mV_0}{\mu}Q_l\left(1+\frac{\mu^2}{2p^2}\right). \end{aligned}$$

For $|\delta_l| \ll 1$, we can expand f_l in first order in δ_l :

$$f_l = \frac{e^{i\delta_l}\sin(\delta_l)}{p} \approx \frac{\delta_l}{p}. \tag{7}$$

Thus :

$$\delta_l = -\frac{mV_0}{\mu p}Q_l\left(1+\frac{\mu^2}{2p^2}\right). \tag{8}$$

2. (a) For $\zeta > 0$, $Q_l(\zeta)$ is positive. Thus when the potential is attractive, $V_0 < 0$, hence $\delta_l > 0$.
   (b) This condition translates to $1/p \gg 1/\mu \Leftrightarrow \mu \gg p$, thus $1 + \mu^2/2p^2 \approx \mu^2/2p^2$. In the series expansion, we can keep only the first term, since the next ones will be at least $1/\zeta^2$ smaller. We obtain :

$$\delta_l = -\frac{mV_0}{\mu p}\cdot\frac{2^{l+1}l!\ p^{2l+2}}{(2l+1)!!\ \mu^{2l+2}} = -\frac{2^{l+1}l!\ mV_0}{(2l+1)!!\mu^{2l+3}}\cdot p^{2l+1}. \tag{9}$$

3. Scattering off a spherical potential

1. Let us start with an impenetrable sphere

$$V(r) = \begin{cases} \infty & \text{for } r < R, \\ 0 & \text{for } r > R. \end{cases} \tag{10}$$

We will use the properties that the wave function must vanish at $r = R$ because the sphere is impenetrable. Therefore we have the following condition on the radial part of the wavefunction :

$$R_l(r)|_{r=R} = 0. \tag{11}$$

For $r > R$, the potential is zero. Hence, we can express R_l as a linear combination of spherical Bessel and von Neumann functions :

$$R_l(r) = \cos\delta_l j_l(kr) - \sin\delta_l y_l(kr). \tag{12}$$

Since we only care about the relative phase shift, we arbitrarily set the amplitude to 1. The boundary condition implies :

$$j_l(kR)\cos\delta_l - y_l(kR)\sin\delta_l = 0. \tag{13}$$

Let us consider the $l = 0$ case (s-wave scattering) specifically. The equation becomes

$$\tan\delta_0 = \frac{\sin kR/kR}{-\cos kR/kR} = -\tan kR. \tag{14}$$

The radial-wave function $R_0(r)$ varies as :

$$R_{l=0}(r) = \frac{\cos\delta_0\sin kr - \sin\delta_0\cos kr}{kr} = \frac{\sin(kr+\delta_0)}{kr}, \tag{15}$$

with $\delta_0 = -kR$. Therefore, if we plot $rR_{l=0}(r)$ as a function of distance r, we obtain a sinusoidal wave that is shifted when compared to the free sinusoidal wave by an amount R.

Let us now study the low- and high-energy limits. For $kR \ll 1$:

$$\begin{aligned} j_l(kr) &\approx \frac{(kr)^l}{(2l+1)!!}, \\ y_l(kr) &\approx -\frac{(2l-1)!!}{(kr)^{l+1}}. \end{aligned} \tag{16}$$

to obtain

$$\tan\delta_l = \frac{-(kR)^{2l+1}}{\left\{(2l+1)\left[(2l-1)!!\right]^2\right\}}. \tag{17}$$

It is therefore justified to ignore δ_l with $l \neq 0$. In other words, we have s-wave scattering only which is actually expected for almost any finite-range potential at low energy.

2. Because $\delta_0 = -kR$ regardless of whether kR is large or small, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2} = R^2 \quad \text{for } kR \ll 1. \tag{18}$$

It is interesting that the total cross section given by

$$\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi R^2. \tag{19}$$

is four times the geometric cross section πR^2. By geometric cross section we mean the area of the disc of radius R that blocks the propagation of the plane wave. Low-energy scattering, of course, means a very large-wavelength scattering, and we do not necessarily expect a classically reasonable result.

4. Scattering off a constant potential

1. Similar to the problem of a square potential in 1D, the idea is to solve the Schrödinger equation on both regions $r < R$ and $r > R$, and connect them by continuity.
   For $r < R$, we have :
$$R_l(r) = A\, j_l(\kappa r), \tag{20}$$
   with $0 < E - V_0 \equiv \frac{\hbar^2\kappa^2}{2m}$. There is no contribution from the von Neumann function, because the wave function must not diverge at $r \to 0$.
   For $r > R$, we have :
$$R_l(r) = B\left[\cos(\delta_l)\, j_l(kr) - \sin(\delta_l) y_l(kr)\right] \tag{21}$$
   In order to find the three unknowns A, B and the scattering phase shift δ_l, we need to match both the wave function and its spatial derivative at the boundary between the two regions, $r = R$:
$$\begin{cases} A\, j_l(\kappa R) = B\left[\cos(\delta_l)\, j_l(kR) - \sin(\delta_l) y_l(kR)\right] \\ A\kappa j_l'(\kappa R) = Bk\left[\cos(\delta_l)\, j_l'(kR) - \sin(\delta_l) y_l'(kR)\right] \end{cases} \tag{22}$$
   To eliminate the normalization constants A and B, we can divide the first equation by the second :
$$\frac{k\, j_l(\kappa R)}{\kappa j_l'(\kappa R)} = \frac{\cos(\delta_l)\, j_l(kR) - \sin(\delta_l) y_l(kR)}{\cos(\delta_l)\, j_l'(kR) - \sin(\delta_l) y_l'(kR)} \tag{23}$$
   We can isolate the scattering phase shift :
$$\tan(\delta_l) = \frac{k j_l'(kR) - \kappa j_l'(\kappa R)\alpha}{k y_l'(kR) - \kappa y_l(kR)\alpha} \tag{24}$$
   with $\alpha \equiv j_l'(\kappa R)/\, j_l(\kappa R)$.
   As $|V_0| \ll E$ and $kR \ll 1$, we also have $\kappa R \ll 1$. Note that we cannot simply use the asymptotic forms of j_l and y_l proved in the first exercise : the derivatives are mixing different orders. This is why we use the identity showed in the hint. Hence $j_l(x) \approx \frac{x^l}{(2l+1)!!}$ for $x \ll 1$, so $j_{l+1}(\kappa R)/\, j_l(\kappa R) = \frac{\kappa R}{2l+3}$ to leading order. We also know that $y_l(x) = -\frac{(2l-1)!!}{x^{l+1}}$ for $x \ll 1$. Therefore the phase shift becomes
$$\begin{aligned} \tan\delta_l &= \frac{(\kappa R)^2 j_l(kR)/(2l+3) - kR\, j_{l+1}(kR)}{(\kappa R)^2 y_l(kR)/(2l+3) - kR y_{l+1}(kR)} \\ &= \frac{(\kappa R)^2 (kR)^l/[(2l+3)!!] - (kR)^{l+2}/[(2l+3)!!]}{-(2l-1)!!(\kappa R)^2/[(2l+3)(kR)^{l+1}] + (2l+1)!!/(kR)^{l+1}]} \\ &\approx (kR)^{2l+1}\frac{(\kappa R)^2 - (kR)^2}{(2l+3)!!(2l+1)!!} \\ &= \frac{(kR)^{2l+3}}{(2l+3)!!(2l+1)!!}\left[\frac{\kappa^2}{k^2} - 1\right]. \end{aligned} \tag{25}$$
   where we ignore the first term in the denominator for $kR \ll 1$. Clearly $l = 0$ dominates, and
$$\tan\delta_0 = \frac{1}{3}(kR)^3\left[\frac{E - V_0}{E} - 1\right] = -\frac{1}{3}(kR)^3\frac{V_0}{E} = -\frac{1}{3}k\frac{2mV_0R^3}{\hbar^2} \approx \delta_0 \approx \sin\delta_0. \tag{26}$$

The total cross section is

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{16\pi}{9}\frac{m^2V_0^2R^6}{\hbar^4}. \tag{27}$$

2. The next most important term is the p-wave. The phase shift is

$$\tan\delta_1 = -\frac{1}{45}(kR)^5\frac{V_0}{E} \approx \delta_1 \approx \sin\delta_1. \tag{28}$$

With just s- and p-waves the differential cross section is

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{k^2}\left|e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1\cos\theta\right|^2 \\ &\approx \frac{1}{k^2}\left(\sin^2\delta_0 + 6\cos(\delta_0-\delta_1)\sin\delta_0\sin\delta_1\cos\theta\right).\end{aligned} \tag{29}$$

which is of the form $\frac{d\sigma}{d\Omega} = A + B\cos\theta$. Since $\delta_1 \ll \delta_0 \ll 1$ we have $\cos(\delta_0-\delta_1) \approx 1$ and

$$\frac{B}{A} = \frac{6\sin\delta_1}{\sin\delta_0} = 6\cdot\frac{3}{45}(kR)^2 = \frac{2}{5}(kR)^2. \tag{30}$$

5. Scattering off $\frac{1}{r^2}$ potential

1. One has to solve the radial Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\left(\frac{l(l+1)+A}{r^2}\right)\right]u_l = Eu_l. \tag{31}$$

For $A = 0$, the known solution involves spherical Bessel functions $j_l(kr)$ and Neumann functions $y_l(kr)$:

$$R_l = \frac{u_l}{r} = a_l j_l(kr) + b_l y_l(kr), \tag{32}$$

with $b_l = 0$ to satisfy the boundary condition $R_l(r \to 0) < \infty$. For $A \neq 0$, we have :

$$R_l = a_l j_n(kr), \tag{33}$$

where n is defined by the quadratic equation $n(n+1) = l(l+1) + A$.

2. The asymptotic form of R_l for large r is :

$$R_l(r) \sim j_n(kr) \sim \frac{1}{r}\sin(kr - \frac{n\pi}{2}) \tag{34}$$

We want to compare this with the definition of the scattering phase shift :

$$R_l(r) \sim \frac{1}{r}\sin(kr - \frac{l\pi}{2} + \delta_l) \tag{35}$$

Solving for n yields :

$$n = \frac{-1 \pm \sqrt{1 + 4(l(l+1)+A)}}{2}. \tag{36}$$

Since we want to let $n = l$ for $A \to 0$, we take the "+" part here. For small A, we use Taylor expansion to find $n \approx l + \frac{A}{2l+1}$.
The phase shift δ_l for small A is approximately :

$$\delta_l = -\frac{\pi A}{2(2l+1)}. \tag{37}$$

The total cross section σ is given by :

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \approx \frac{\pi^3 A^2}{k^2}\sum_{l=0}^{\infty}\frac{1}{2l+1} \to \infty. \tag{38}$$

The series of inverses of odd integers diverges, hence the total scattering cross section also diverges. This happens because the $1/r^2$ potential is too long range : it does not decay fast enough as $r \to \infty$. We need at least a cubic decay $V \sim 1/r^3$ to have a finite cross section.