QUANTUM PHYSICS III

Solutions to Problem Set 10 19 November 2024

1. Applicability condition of the first Born approximation

1. First, we rewrite the difference $\Psi_{in}(\mathbf{x}) - \Psi_0(\mathbf{x})$ by using the relation between $\Psi_{in}(\mathbf{x})$ and $\Psi_0(\mathbf{x})$,

$$
\begin{aligned}
\Psi_0(\mathbf{x}) &= \Psi_{in}(\mathbf{x}) + \int d^3\mathbf{p} \langle \mathbf{x}|\hat{G}(E_p + i\epsilon)\hat{V}|\mathbf{p}\rangle\langle\mathbf{p}|\Psi_{in}\rangle \\
&= \Psi_{in}(\mathbf{x}) + \int d^3\mathbf{p} d^3\mathbf{x}' d^3\mathbf{x}'' \langle \mathbf{x}|\hat{G}(E_p + i\epsilon)|\mathbf{x}'\rangle\langle\mathbf{x}'|\hat{V}|\mathbf{x}''\rangle\langle\mathbf{x}''|\mathbf{p}\rangle\langle\mathbf{p}|\Psi_{in}\rangle \qquad (1) \\
&= \Psi_{in}(\mathbf{x}) + \int d^3\mathbf{x}' G(E_p + i\epsilon, \mathbf{x}, \mathbf{x}')V(\mathbf{x}')\Psi_{in}(\mathbf{x}') \ .
\end{aligned}
$$

To the leading order in perturbation theory, in the last line one can replace G by G_0, since the difference between the free and the full Green functions provides the next-order correction to $\Psi_0(\mathbf{x})$. Since

$$\Psi_{in}(\mathbf{x}) \sim e^{i\mathbf{p}\cdot\mathbf{x}} \ , \tag{2}$$

the inequality $|\Psi_{in}(\mathbf{0}) - \Psi_0(\mathbf{0})| \ll |\Psi_{in}(\mathbf{0})|$ is rewritten as

$$\left|\int d^3\mathbf{x}' G_0(E_p - i\epsilon, \mathbf{0}, \mathbf{x}')V(\mathbf{x}')\Psi_{in}(\mathbf{x}')\right| \ll 1 \ . \tag{3}$$

Recall that

$$G_0(E_p + i\epsilon, \mathbf{x}, \mathbf{x}') = \frac{m}{2\pi}\frac{e^{i\sqrt{2mE_p}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \ . \tag{4}$$

Then, if $V(\mathbf{x})$ contains only radial dependence on $\mathbf{x}$, one can perform explicitly the integration over angular variables in eq. (3) :

$$
\begin{aligned}
\left|\int d^3\mathbf{x}' G_0(E_p - i\epsilon, \mathbf{0}, \mathbf{x}')V(\mathbf{x}')\Psi_{in}(\mathbf{x}')\right| &= m\left|\int_0^\infty dr\ r^2 d\cos\theta \frac{e^{i\sqrt{2mE_p}r}}{r}V(r)e^{ipr\cos\theta}\right| \\
&= m\left|\int_0^\infty dr\ r e^{i\sqrt{2mE_p}r}V(r)\frac{1}{ipr}\left(e^{ipr} - e^{-ipr}\right)\right| \\
&\leqslant \frac{m}{p}\left|\int_0^\infty drV(r)\left(1 - e^{2ipr}\right)\right| \ .
\end{aligned} \tag{5}
$$

Hence, eq. (3) takes the form

$$\frac{m}{p}\left|\int_0^\infty drV(r)\left(1 - e^{2ipr}\right)\right| \ll 1 \ . \tag{6}$$

2. Consider the square well potential

$$V(r) = \begin{cases} -V_0\,, & r < R\,, \\ 0\,, & r > R\,. \end{cases} \tag{7}$$

In the slow scattering regime, $pR \ll 1$, we can expand the exponent in eq. (6) to the first order in pR to obtain

$$\frac{m}{p}\left|V_0 \int_0^R dr\, 2ipr\right| \sim mV_0R^2 \ll 1\,. \tag{8}$$

3. Since $\sigma \sim 4\pi|f|^2$, and $f \sim \frac{m}{2\pi}V_0\frac{4\pi}{3}R^3 \sim mV_0R^3$, we have by the virtue of eq. (8)

$$\sigma \sim 4\pi m^2V_0^2R^6 = 4\pi(mV_0R^2)^2 \cdot R^2 \ll 4\pi R^2\,. \tag{9}$$

4. In the regime of fast scattering, $pR \gg 1$, the exponent in eq. (6) is a rapidly oscillating function which gives no overall contribution to the integral. Hence, the applicability condition becomes $\frac{m}{p}V_0R \ll 1$, or

$$mV_0R^2 \ll pR\,. \tag{10}$$

Note that this requirement is much weaker than the condition (8) for slow particles. This is consistent with expectations, since for a given potential the Born approximation is supposed to work better as the energy of the scattered particles increases.

5. Applying the condition (6) to the Yukawa potential,

$$V(r) = \frac{\alpha}{r}e^{-\mu r}, \tag{11}$$

we have

$$\frac{m}{p}\left|\int_0^\infty dr\frac{\alpha}{r}e^{-\mu r}\sin pr\right| = \left|\frac{m\alpha}{p}\arctan\frac{p}{\mu}\right| \sim \begin{cases} \left|\frac{m\alpha}{\mu}\right| \ll 1, & p \ll \mu, \\ \left|\frac{m\alpha}{p}\right| \ll 1, & p \gg \mu. \end{cases} \tag{12}$$

2. Towards the inverse scattering problem

1. Let R be the characteristic size of the potential $V(r)$. The scattering amplitude at zero momentum transfer f_0 is given by

$$f_0 \approx -\lim_{q\to 0}\frac{2m}{q}\int_0^R dr\, rV(r)\sin qr = -2m\int_0^R dr\, r^2V(r)\,. \tag{13}$$

Next, we compute the amplitude assuming that $qR \ll 1$, this gives

$$\begin{aligned} f(q) &\approx -\frac{2m}{q}\int_0^R dr\, rV(r)\left(qr - \frac{1}{6}(qr)^3\right) \\ &= f_0 - f_0\frac{(qR)^2}{10}\,. \end{aligned} \tag{14}$$

From here, one can extract the size R as

$$R^2 \approx \frac{10}{f_0}\frac{f_0 - f(q)}{q^2} \approx \frac{10}{f_0}\frac{|f'(q)|}{q} \approx \frac{10C}{f_0} \ . \tag{15}$$

2. Assume that at small distances the potential exhibits the power-like behavior, $V(r) \sim r^n$. Then,

$$f(q) \approx -\frac{2m}{q}\int_0^R dr\ r^{n+1}\sin qr = -\frac{2m}{q^{n+3}}\int_0^{qR} dy\ y^{n+1}\sin y\ , \tag{16}$$

where we denoted $y = qr$. If $qR \gg 1$, one can replace the upper limit of integration in the r.h.s. by infinity, hence at the large momentum transfers

$$f(q) \sim \frac{1}{q^{n+3}}\ . \tag{17}$$

Comparing this with the given data gives $\frac{N}{2} = n + 3$, and

$$V(r) \sim r^{\frac{N}{2}-3}\ , \quad r \to 0\ . \tag{18}$$

3. Truncation of the Coulomb potential

1. For the exponential shielding we find

$$\begin{aligned} f_1(\mathbf{p} \to \mathbf{p}') &= -\frac{2m}{q}\int_0^\infty dr\ r\frac{\alpha}{r}e^{-\frac{r}{\rho}}\sin qr \\ &= -\frac{2m\alpha}{2iq}\int_0^\infty dr\left(e^{r\left(iq-\frac{1}{\rho}\right)} - e^{r\left(iq-\frac{1}{\rho}\right)}\right) \\ &= -\frac{i\alpha m}{q}\left(\frac{1}{iq+\frac{1}{\rho}} + \frac{1}{iq-\frac{1}{\rho}}\right) = -\frac{2\alpha m}{q^2+\rho^{-2}}\ . \end{aligned} \tag{19}$$

Note that the limit $\rho \to \infty$ is well-defined unless $q = 0$. Hence, except for the forward scattering cone, the amplitude for the exponentially truncated potential is ρ-independent for ρ large enough. In fact, in this limit f_1 reproduces the correct scattering amplitude for the Coulomb potential.

2. Evaluation of the sharp cutoff gives,

$$f_2(\mathbf{p} \to \mathbf{p}') = -\frac{2m}{q}\int_0^\rho dr\ r\frac{\alpha}{r}\sin qr = -\frac{2m\alpha}{q^2}(1-\cos q\rho)\ . \tag{20}$$

This is again a well-defined expression but the one which has no limit at $\rho \to \infty$. We conclude that the answer for the scattering amplitude depends on *how* the truncation of the Coulomb potential is made. At first sight this fact seems distressing, but let us see how the truncation affects the quantities one can actually observe in experiment.

3. Let us assume that $q\rho \gg 1$, or

$$\rho \gg \frac{1}{2p \sin \frac{\theta}{2}} , \tag{21}$$

which can always be justified unless $\theta = 0$. Then, the ratio of the amplitudes (20) and (19) averaged over the range of the scattering angles from θ to $\theta + \Delta\theta$ is equal to

$$\frac{1}{\Delta\theta} \int_{\theta}^{\theta+\Delta\theta} d\theta' \left| \frac{f_2(\theta')}{f_1(\theta')} \right| = 1 - \frac{1}{\Delta\theta} \int_{\theta}^{\theta+\Delta\theta} d\theta' \cos\left(2p\rho \sin \frac{\theta}{2}\right) . \tag{22}$$

The integrand in the second term is a rapidly oscillating function that is integrated to zero provided that

$$2p\rho \left(\sin \frac{\theta + \Delta\theta}{2} - \sin \frac{\theta}{2} \right) \gg 2\pi . \tag{23}$$

For $\theta \neq \pi$, one can expand the sin to the first power in $\Delta\theta$ to obtain from eq. (23)

$$2p\rho \frac{1}{2} \frac{\Delta\theta}{2} \cos \frac{\theta}{2} \gg 2\pi \quad \Rightarrow \quad \rho \gg \rho_0 = \frac{4\pi}{p \Delta\theta \cos \frac{\theta}{2}} . \tag{24}$$

For $\theta = \pi$, we expand the sin to the second power in $\Delta\theta$ and find

$$2p\rho \frac{\Delta\theta^2}{8} \gg 2\pi \quad \Rightarrow \quad \rho \gg \rho_0 = \frac{8\pi}{p \Delta\theta^2} . \tag{25}$$

4. Given the in wave packet $\Psi_{in}(\mathbf{p})$, the out wave function $\Psi_{out}(\mathbf{p})$ is evaluated in the first Born approximation as

$$\begin{aligned} \Psi_{out}(\mathbf{p}) &= \Psi_{in}(\mathbf{p}) + \frac{i}{2\pi m} \int d^3\mathbf{p}' \delta(E_p - E_{p'}) f(\mathbf{p} \to \mathbf{p}') \Psi_{in}(\mathbf{p}') \\ &= \Psi_{in}(\mathbf{p}) + \frac{ip}{2\pi} \int d\Omega_{p'} f(\mathbf{p} \to \mathbf{p}') \Psi_{in}(\mathbf{p}') . \end{aligned} \tag{26}$$

In this expression, the first term represents the unscattered incident wave, and to avoid seeing this term one normally restricts the measurements to non-forward directions. Then, the difference between the amplitudes f_1 and f_2 contains the rapidly oscillating term

$$\cos q\rho = \cos\left(2p\rho \sin \frac{\theta}{2}\right) , \quad \theta \neq 0 , \tag{27}$$

which, for ρ exceeding the size of the initial wave packet, integrates out to zero, and makes no contribution to $\Psi_{out}(\mathbf{p})$. Thus, the difference between the two methods of screening the Coulomb potential has no observable effect.