QUANTUM PHYSICS III

Problem Set 6 15 October 2024

1. Restoration of symmetry in the double-well potential

Consider the symmetric double-well potential of the form

$$V(x) = V_0 (x - x_0)^2 (x + x_0)^2 \ , \quad V_0 > 0 \ . \tag{1}$$

We are interested in the time evolution of the wave function $\Phi(x,t)$ of a particle of mass m, whose energy is well below the barrier separated the wells. Specifically, let the particle be initially localized, say, in the left well,

$$\Phi(x, 0) = \psi_0(x) \ , \tag{2}$$

where $\psi_0(x)$ denotes the bound state of the left well, and we assume the energy of this state to be much smaller than the height of the barrier. The wave packet (2) breaks the parity symmetry of the system. Recall, however, that due to the tunneling phenomenon, the probability to detect the particle in the right well is nonzero at all $t > 0$, and if we wait *sufficiently* long, we should be able to find the particle in either well with almost equal average probabilities. So, the symmetry gets restored in the limit $t \to \infty$, and this exercise is suggested to demonstrate this explicitly.

1. Write the probability $P(x,t)$ to find the particle at the position x and at the time t.
2. Find the explicit expression for the probability $P(t)$ to find the particle in the right well (that is, at $x > 0$).
3. Compute the average probability to find the particle in the right well in the limit of large detection time T :

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T P(t) dt \ . \tag{3}$$

2. WKB spectrum of the Hydrogen atom

Electron levels in the Hydrogen atom are characterized by three quantum number : n_r (the radial number), l (the angular momentum number), and m (the magnetic number). To find the energies of the levels, we consider the Coulomb potential supplemented by a centrifugal term of the form

$$V(r) = -\frac{1}{a_0 M r} + \frac{(l + 1/2)^2}{2Mr^2} \ , \tag{4}$$

where a_0 is the Bohr radius, M is the electron mass, and we put $\hbar = 1$.

1. Using the WKB approach, find the energy spectrum of an electron in the potential (4). Compare with the exact answer.
   Hint : $\int_{x_1}^{x_2} dx \sqrt{\left(1 - \frac{x_1}{x}\right)\left(\frac{x_2}{x} - 1\right)} = \frac{\pi}{2}(x_1 + x_2 - 2\sqrt{x_1 x_2})$.
2. Compute the energy of the ground state, in eV.
3. What is the degeneracy of the n'th energy level ?

3. Classical scattering on a Coulomb potential

Consider a constant flux of non-interacting particles (i.e. a constant number of n particles per area and time) of mass m with fixed energy and direction approaching a central potential $U(r)$ (a scattering center).

1. Show that the orbit equation for each individual particle is given by (see figure 1)

$$\phi(r) = \int_{\infty}^{r} \frac{L/r'^2 dr'}{\sqrt{2m(E - U(r')) - L^2/r'^2}} , \tag{5}$$

   with E the energy and L the angular momentum. For the Coulomb potential $U(r) = \alpha/r$ with $\alpha \in$ Reals and for $E > 0$ this is a scattering orbit (a hyperbola).
2. Use the previous equation to determine the deflection angle θ for a particle starting at $r = \infty$ and going back to $r = \infty$.

   Hint : Use the formula

$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{x\sqrt{b^2 - 4ac}} . \tag{6}$$

3. Replace the constants of motion (E, L) by (E, b), with b the impact parameter, i.e. the normal distance between the asymptote of the incident particle and the scattering center at $r = 0$.

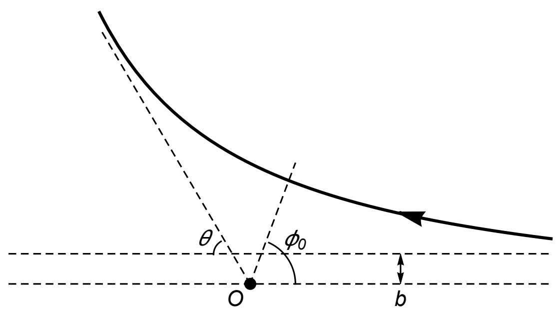

FIG. 1 – The Coulomb potential

4. Determine the number of particles dN per area and per time in a ring between b and $b + db$. If there is a one-to-one functional relation $b(\theta)$ between b and the scattering angle, then dN is at the same time the number of particles that is scattered in an

angle between θ and $\theta + d\theta$. Use this to show that the differential cross section for a Coulomb scattering (i.e. the Rutherford scattering formula) is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\frac{1}{\sin^4\frac{\theta}{2}} \,. \tag{7}$$

5. Show that the total cross section is infinite. Interpret the result.

4. Differential cross section transformation

Consider a particle of mass m_1 scattering off a target particle of mass m_2 in the non-relativistic limit.

1. Show that the relation between the differential cross section in the laboratory frame at a given lab angle $\theta_{\rm LAB}$ and the differential cross section in the center of mass frame at the corresponding angle $\theta_{\rm CM}$ can be written as

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm LAB} = \frac{\left(1 + 2\lambda\cos\theta_{CM} + \lambda^2\right)^{3/2}}{|1 + \lambda\cos\theta_{CM}|}\left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} \tag{8}$$

with $\lambda = m_1/m_2$ the mass ratio of the two particles.

Hint : Show that the relation between $\cos\theta_{\rm LAB}$ and $\cos\theta_{\rm CM}$ is given by

$$\cos\theta_{\rm LAB} = \frac{\cos\theta_{\rm CM} + \lambda}{(1 + 2\lambda\cos\theta_{\rm CM} + \lambda^2)^{1/2}} \,. \tag{9}$$

5. Interaction picture

Consider a system with the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ is the free Hamiltonian and $\hat{V}$ is the interaction. Define the interaction picture for states and operators via the relations

$$\begin{aligned} \Psi_I(t) &= \hat{U}_0^\dagger(t)\Psi_S(t) \,, \\ \hat{A}_I(t) &= \hat{U}_0^\dagger(t)\hat{A}_S\hat{U}_0(t) \,, \end{aligned} \tag{10}$$

where $\hat{U}_0(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}$, and the subscript S denotes quantities in the Schrodinger picture.

1. Find the relation between the states and operators in the interaction and Heisenberg pictures.
2. Show that the evolution of the wave function in the interaction picture is described by the interaction term $\hat{V}$ in the same picture, i.e.

$$-\frac{\hbar}{i}\frac{d}{dt}\Psi_I(t) = \hat{V}_I\Psi_I(t) \,. \tag{11}$$

3. Express the evolution operator in the interaction picture $\hat{U}_I(t)$ through $\hat{U}(t)$ and $\hat{U}_0(t)$. Find a differential equation which $\hat{U}_I(t)$ obeys and determine the initial condition for it.