QUANTUM PHYSICS III

Problem Set 4 1 October 2024

1. The one-turning point problem

In the classically forbidden region $x < x_0$, the LO WKB wave function is given by

$$\psi(x) = \frac{C}{2\sqrt{|p(x)|}} \exp\left(-\frac{1}{\hbar}\int_x^{x_0} |p(x')|dx'\right) . \tag{1}$$

1. Find the expression for this wave function in the classically allowed region $x > x_0$.

2. Quantization rule in a half-space

The Bohr-Sommerfeld quantization rule

$$\oint p dx = 2\pi\hbar\left(n + \frac{1}{2}\right) \tag{2}$$

was derived under the assumption that there are two turning points beyond which, in the classically forbidden regions, the wave function ψ falls off exponentially fast. This amounts to imposing the boundary conditions $\psi(\pm\infty) = 0$ that are exact in any order in $\hbar$. What if a given physical problem requires other boundary data ? For example, consider the following conditions,

$$\psi(\infty) = 0\ , \quad \psi(x) = 0\ , \quad x \leqslant 0\ . \tag{3}$$

They imply that the potential in the problem is supplemented by an infinite wall at $x = 0$ beyond which no wave function can penetrate (see figure 1).

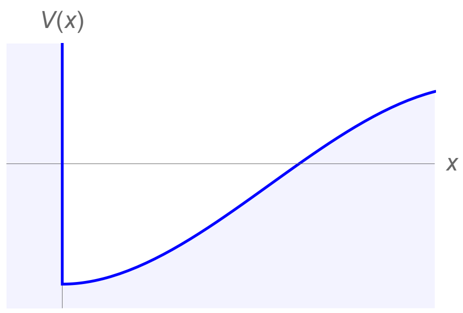

FIGURE 1 – The infinite wall potential

1. Derive the quantization rule for bound states in this type of potentials.

The rule you have found has a natural application to computation of energy levels of three-dimensional systems possessing spherical symmetry. Indeed, the potential of the system in this case depends only on the radial coordinate r. If the potential is regular at $r = 0$, then for the wave functions with zero orbital momentum the problem reduces to solving the one-dimensional Schroedinger equation with $V = V(r)$ at $r > 0$, and $V = \infty$ at $r = 0$.

2. In the LO WKB approach, find energy levels of bottomonium — a pair of nonrelativistic quark and antiquark with masses $m_c = 4.98\,GeV$ placed in a linear potential $V = V_0 + kr$, with $V_0 = 5\,MeV$ and $k = 0.8\,GeV^2$.

3. WKB spectrum of the Harmonic oscillator

Consider the particle of mass m, moving in the potential $V(x) = \frac{1}{2}m\omega^2 x^2$.

1. Find the LO WKB energy levels of the particle. Compare with the exact answer.
2. Find for which values of E you can trust the results according to the LO WKB applicability conditions.

4. WKB spectrum in power-like potential

At large n, the qualitative dependence of the energy E_n of the n'th WKB bound state is of the form

$$E_n \sim n^\beta \ , \quad n \gg 1 \ , \tag{4}$$

where the exponent β is determined by the potential. For example, from the previous exercise we know that for the harmonic oscillator $\beta = 1$. To analyze the range of possible values of β, consider the particle of mass m, confined in the potential

$$V(x) = V_0 \left| \frac{x}{x_0} \right|^\alpha \ , \quad \alpha > 0 \ . \tag{5}$$

1. Find the LO WKB spectrum of the particle. Check that for $\alpha = 2$ the answer reduces to the spectrum of the harmonic oscillator.
   Hint : Use the formula

   $$\int_0^1 \sqrt{1 - y^\alpha} dy = \frac{\sqrt{\pi}}{2} \frac{\Gamma(1 + \frac{1}{\alpha})}{\Gamma(\frac{3}{2} + \frac{1}{\alpha})} \ , \quad \alpha > 0 \ , \tag{6}$$

   where $\Gamma(z)$ is the Gamma function.
2. Plot the function $\beta = \beta(\alpha)$, where β is defined in eq. (4). What happens when $\alpha \to \infty$?

5. The multifold one-turning point problem

Consider the wave function continued from the classically forbidden region $x > 0$ to the classically allowed region $x < 0$ through the turning point of multiplicity $2k + 1$, with k an integer number (see figure 2),

$$V(x) - E \sim x^{2k+1}\ , \quad |x| \ll 1\ . \tag{7}$$

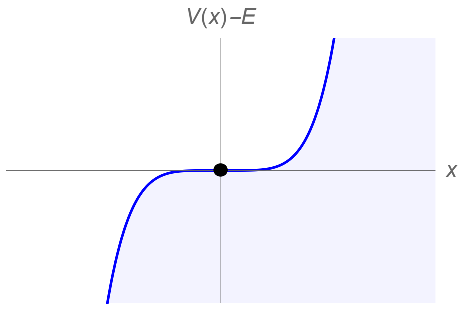

FIGURE 2 – The turning point of high multiplicity

1. Find the LO WKB wave function of a particle at $x < 0$, if at $x > 0$ it is given by

$$\psi(x) = \frac{C}{2\sqrt{|p|}} \exp\left(-\frac{1}{\hbar}\left|\int_0^x p\, dx\right|\right)\ , \quad x > 0\ . \tag{8}$$