QUANTUM PHYSICS III

Problem Set 3 24 September 2024

1. On validity of the leading-order (LO) WKB approximation

Consider the semiclassical expansion of the wave function

$$\psi = e^{\frac{i}{\hbar}S} \ , \quad S = S_0 + \frac{\hbar}{i} S_1 + \left(\frac{\hbar}{i}\right)^2 S_2 + \dots \ . \tag{1}$$

It is shown in the lecture notes that one necessary applicability condition of the LO semi-classical approximation

$$\psi \approx e^{\frac{i}{\hbar}(S_0 + \frac{\hbar}{i} S_1)} \tag{2}$$

is formulated as $|\lambda\!\!\!^{-}{}'| \ll 1$, where $\lambda\!\!\!^{-} = \hbar/p$ is the de Broglie wave length. However, this requirement is by no means *sufficient* to ensure the validness of the LO approximation, and in this exercise we have a look at other necessary conditions.

1. Expanding the Schrödinger equation for ψ up to the second order in $\hbar$, find S_2' in terms of S_0' and S_1'. Rewrite it through the momentum p and its derivatives ; through the energy of the particle E, the potential V and its derivatives.
2. Show that for the approximation (2) to hold, one must require $|\hbar^{n-1} S_n| \ll 1$ for all $n \geqslant 2$.
3. Show that the first condition in this chain, $|\hbar S_2| \ll 1$, follows from $|\lambda\!\!\!^{-}{}'| \ll 1$ and $\int^x |\lambda\!\!\!^{-}{}'^2 \lambda\!\!\!^{-}{}^{-1}| \, dx \ll 1$.
4. From the condition $|\lambda\!\!\!^{-}{}'| \ll 1$ obtain the following inequality,

$$\left| \frac{\delta A}{T_{kin}} \right| \ll 1 \ , \tag{3}$$

where δA is a work done by a force $F = -V'$ on a distance $\lambda\!\!\!^{-}$, and T_{kin} is a kinetic energy of the particle.

2. On accuracy of the LO WKB approximation

Consider the particle of unit mass and with the energy V_0, moving in the potential

$$V(x) = \begin{cases} 0 \ , & x < 0 \ , \\ V_0 \sqrt{\dfrac{x}{x_0}} \ , & x > 0 \ . \end{cases} \tag{4}$$

1. Find the LO WKB wave function of the particle in the region $x > x_0$.
2. Find how small (or large) one should take V_0 to be sure that the LO approximation of the wave function is accurate to 1 percent for all $x > 2x_0$.

3. On asymptotics of the potential in the WKB approximation

Consider the particle of zero energy, moving in the potential $V(x)$ shown schematically on figure 1. Assume that $V(x)$ approaches a constant negative value at $x \to -\infty$, and that its behavior at large positive x is of the form

$$V(x) \sim x^{-n}\,, \quad n > 0\,, \quad x \to \infty\,. \tag{5}$$

1. Give a constraint on possible values of n, that ensures the validness of the LO WKB approximation of the decaying wave function in the limit $x \to \infty$.

Suppose now that the asymptotics of the potential at large positive x is

$$V(x) \sim \left(\frac{\log x}{x}\right)^2\,. \tag{6}$$

2. Is it legitimate to use WKB theory to predict the large-x behavior of the decaying wave function ?

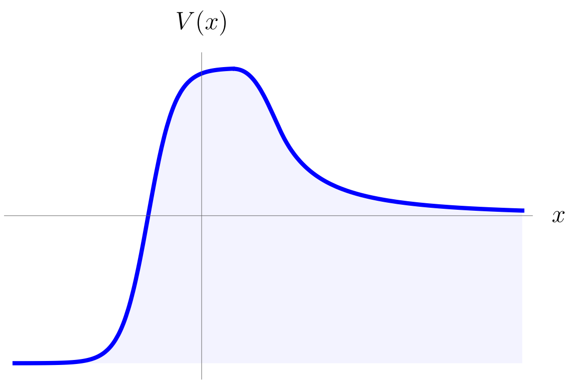

FIGURE 1 – The potential

4*. WKB expansion beyond the LO

The semiclassical approximation of the amplitude (2) can be continued beyond the LO by successive calculation of higher-order terms in the expansion (1). These terms exhibit some interesting properties which, as we will see later, are important when computing the energy levels of the particle.

1. Obtain an iterative expression for S'_n through $S'_0, ..., S'_{n-1}$.
2. Show that all *odd* terms S'_{2k+1} are
   (a) real (and, hence, do not contribute to the phase of the wave function),
   (b) total derivatives.