
QUANTUM PHYSICS III

Problem Set 2

17 September 2024

1. Classical limit of the harmonic oscillator

Given an operator $\hat{A}$, denote by $\langle \hat{A} \rangle_\alpha$ the average $\langle \hat{A} \rangle_\alpha = \langle \alpha | \hat{A} | \alpha \rangle$, with $|\alpha\rangle$ the coherent state of the harmonic oscillator. Recall also that $(\Delta \hat{A}_\alpha)^2 = \langle \hat{A}^2 \rangle_\alpha - \langle \hat{A} \rangle_\alpha^2$.

1. Calculate $\langle \hat{H} \rangle_\alpha$, $\Delta \hat{H}_\alpha$, $\langle \hat{N} \rangle_\alpha$, $\Delta \hat{N}_\alpha$, $\langle \hat{X} \rangle_\alpha$, $\Delta \hat{X}_\alpha$, $\langle \hat{P} \rangle_\alpha$, $\Delta \hat{P}_\alpha$. What is the product $\Delta \hat{X}_\alpha \cdot \Delta \hat{P}_\alpha$? Write the validity conditions of the classical approach. What do they impose on $|\alpha|$?
2. As an application, consider a classical pendulum of length $l = 0.1 \text{ m}$ and mass $m = 1 \text{ kg}$ placed in the gravitational field of the Earth, that performs a periodic motion with the amplitude $x_M = 1 \text{ cm}$. One can treat it as a coherent state $|\alpha\rangle$ of the quantum harmonic oscillator. Compute the numerical values of $|\alpha|$, $\Delta \hat{X}_\alpha$, $\Delta \hat{P}_\alpha$, $\Delta \hat{H}_\alpha$ and $\langle \hat{H} \rangle_\alpha$ and check the validity of the classical description.

2. Squeezed states

From the previous exercise we learned that the coherent states of the harmonic oscillator minimize the uncertainty relation. Are there other states whose behavior resembles classical? Let us consider the following state $|\Psi_\lambda\rangle$ (at $t = 0$):

$$|\Psi_\lambda\rangle \text{ is such that } \langle x | \Psi_\lambda \rangle \equiv \Psi_\lambda(x) = C \Psi_\alpha(\lambda x), \quad (1)$$

with λ some real constant, C a normalization factor and $|\Psi_\alpha\rangle \equiv |\alpha\rangle$ a coherent state of the harmonic oscillator. The states of the form (1) are called the *squeezed* states. The name comes from the fact that for such states the dispersions of the coordinate and momentum operators can be less than those for the ground state of the harmonic oscillator. Given below is a chain of exercises aiming to unveil some properties of the squeezed states.

1. Determine the normalization factor C .
2. For the state $|\Psi_\lambda\rangle$ find $\langle \hat{X} \rangle$, $\langle \hat{P} \rangle$, $\Delta \hat{X}$, $\Delta \hat{P}$. What is the value of $\Delta \hat{X} \cdot \Delta \hat{P}$?
3. Find a *unitary* operator $\hat{S}_\lambda$ whose action on $|\Psi_\alpha\rangle$ gives $|\Psi_\lambda\rangle$.
Hint : Recall that the coordinate shift $\Psi(x) \Rightarrow \Psi(x + a)$ can be achieved by the action of a translation operator of the form $\hat{T}(a) = e^{\frac{i}{\hbar} \hat{P} a}$.
4. Show that the time-evolved state $|\Psi_\lambda(t)\rangle$ can be expressed as

$$|\Psi_\lambda(t)\rangle = \hat{S}_\lambda(-t) |\Psi_\alpha(t)\rangle, \quad (2)$$

where $\hat{S}_\lambda(-t)$ is the operator $\hat{S}_\lambda$ in the Heisenberg picture.

5. Express $\hat{S}_\lambda(-t)$ in terms of the creation and annihilation operators $\hat{a}$, $\hat{a}^\dagger$ of the harmonic oscillator.

6. Now compute the expressions $\widehat{S}_\lambda^\dagger(-t) \widehat{a} \widehat{S}_\lambda(-t)$ and $\widehat{S}_\lambda^\dagger(-t) \widehat{a}^\dagger \widehat{S}_\lambda(-t)$.
7. Using the results of the p. 6, find the dispersions $\Delta\widehat{X}$ and $\Delta\widehat{P}$ for the state $|\Psi_\lambda(t)\rangle$ at $t \geq 0$. Check that they match with the results of the p. 2 at $t = 0$. Describe the time evolution of $\Delta\widehat{X}$ and $\Delta\widehat{P}$. Does the state $|\Psi_\lambda(t)\rangle$ minimize the uncertainty relation for the pair of operators $\widehat{X}, \widehat{P}$ at arbitrary $t > 0$? At some specific $t > 0$?
8. Find a time-dependent canonical transformation of the operators $\widehat{X}$ and $\widehat{P}$ such that the transformed operators $\widehat{X}', \widehat{P}'$ minimize the uncertainty relation in the state $|\Psi_\lambda(t)\rangle$ at any $t > 0$.