
QUANTUM PHYSICS III

Problem Set 13

17 December 2024

1. Non-relativistic limit of the Dirac equation

Previously we discussed the Dirac equation for the particle moving in an external electromagnetic field. Let us now investigate the non-relativistic limit of this equation which we expect to be of the form of the Schroedinger equation, and the form of the leading-order relativistic corrections. To simplify the treatment, consider the spherically-symmetric static electric field for which

$$\vec{A} = 0, \quad e\Phi = V(r). \quad (1)$$

Then, the Dirac equation reads as follows,

$$H\Psi = \mathcal{E}\Psi, \quad H = c\vec{\alpha} \cdot \vec{p} + \beta mc^2 + V(r), \quad \mathcal{E} = E + mc^2, \quad (2)$$

where $\Psi = (\phi, \chi)^T$ is the Dirac spinor and we put $\hbar = 1$. The non-relativistic limit implies

$$\frac{p^2}{2m} \ll mc^2, \quad V(r) \ll mc^2, \quad (3)$$

hence the expansion in eq. (2) can be performed with respect to the speed of light c . We will work in the Dirac representation of the Dirac matrices.

1. Find the expression for χ through ϕ with the accuracy $O(1/c^3)$. Using it, obtain an equation on ϕ with the accuracy $O(1/c^2)$.
2. Rewrite the equation for ϕ in the form

$$E\phi + \frac{p^2}{2m} \frac{E}{2mc^2} \phi = \tilde{H}\phi \quad (4)$$

with $\tilde{H}$ some hermitian operator. To simplify the l.h.s. of eq. (4), introduce a new variable

$$\xi = \sqrt{1 + \frac{p^2}{2m} \frac{1}{2mc^2}} \phi \quad (5)$$

and rewrite eq. (4) in the form of the Schroedinger equation

$$H_{eff}\xi = E\xi. \quad (6)$$

3. Extract the part of H_{eff} which does not depend on c . It should give you the usual Schroedinger equation on ξ .
4. Now extract the $O(c^{-2})$ -term in H_{eff} and bring it to the form

$$\underbrace{-\frac{1}{2mc^2} \frac{(p^2)^2}{4m^2}}_{\text{rel. correction to the energy}} + \underbrace{\frac{\hbar \vec{\sigma}}{2m \cdot 2mc^2} \cdot (\vec{\nabla} V \times \vec{p})}_{\text{spin-orbital interaction}} + \underbrace{\frac{\hbar^2}{8m^2 c^2} \Delta V}_{\text{the Darwin term}}. \quad (7)$$

5. Rewrite the expression (7) for the Coulomb potential

$$V(r) = -\frac{Ze^2}{r} . \quad (8)$$

Indication: recall that the orbital momentum of the particle is $\vec{L} = [\vec{x}, \vec{p}]$, and the vector of its spin is $\vec{s} = \hbar\vec{\sigma}/2$.

2. Zitterbewegung

The physical meaning of the first two relativistic corrections in eq. (7) is quite clear: the first comes from the expansion of the relativistic energy of the particle, $E^2 = m^2c^4 + (p^2/2m)^2$, and the second represents the spin-orbital coupling. But what is the Darwin term? It can be attributed to a peculiar motion of a Dirac particle called *Zitterbewegung*. One way to see it is by using the Heisenberg equation of motion. Let us take the free particle Dirac Hamiltonian H_D , then

$$i\hbar \frac{dO}{dt} = [O, H_D] , \quad (9)$$

where O is some operator (an observable).

1. By taking $O = \ddot{\vec{x}}$ in eq. (9), obtain an equation on the observable $\ddot{\vec{x}}$. Find the general solution of this equation. Integrating it, obtain an expression for the coordinate $\vec{x}$.
2. Determine all arbitrary constants in the expression for $\vec{x}$ found above by comparing its derivatives with the Heisenberg equations written for $\vec{x}$ and $\dot{\vec{x}}$.
3. What is the behavior of the coordinate $\vec{x}$? Does it oscillate and if so, what is the frequency of the oscillations (the formula for it and the numerical value)?
4. If the Dirac particle is moving in a (weak) external electric potential V , the average value of the potential felt by the particle is given by

$$\langle V \rangle = \frac{1}{2} \langle x^i x^j \rangle \frac{\partial^2 V}{\partial x^i \partial x^j} . \quad (10)$$

Making use of the explicit formula for $\vec{x}$, find $\langle V \rangle$. Compare with the third term of eq. (7).

Hint: What is the value of $\langle x^i x^j \rangle$ in the isotropic background?