
QUANTUM PHYSICS III

Problem Set 12

3 December 2024

1. General solution of the Dirac equation

Due to linearity, the Dirac equation

$$-\frac{\hbar}{i} \frac{\partial \Psi_D}{\partial t} = H_D \Psi_D \quad (1)$$

admits a decomposition of its general solution to the plane-wave functions,

$$\Psi_D = e^{\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{x} - \omega_P t)} u_P, \quad (2)$$

where u_P is some function of the momentum $\mathbf{p}$.

1. Rewrite eq. (1) as an equation on u_P .
2. Find the necessary and sufficient condition for this equation to have a non-zero solution. What is the physical meaning of this condition?
3. Find the general solution of the equation above.
Hint: At this point it is convenient to remember that u_P is a column $(\phi_P, \chi_P)^T$ of two-component functions ϕ_P and χ_P .
4. Rewrite the general solution in the non-relativistic limit $p \ll m$.

2. Properties of the Dirac matrices

Recall that the Dirac Hamiltonian H_D is given by the following 4×4 matrix,

$$H_D = \sum_{i=1}^3 \alpha_i p_i + \beta m, \quad (3)$$

where

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (4)$$

Here σ_i are the Pauli matrices, and I denotes the 2×2 identity matrix. The matrices α_i and β obey certain relations which, however, do not specify them fully, hence the choice (4) is not unique.

1. Given α_i, β , one can define new matrices α'_i, β' via

$$\alpha'_i = U \alpha_i U^{-1}, \quad \beta' = U \beta U^{-1}, \quad (5)$$

where U is a unitary but otherwise arbitrary 4×4 matrix. Show that α'_i, β' form an appropriate set of matrices provided that α_i, β do.

2. Find the matrix U that transforms β , given in eq. (4), into $\beta' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Find α'_i corresponding to this transformation. This choice of the Dirac matrices is called the Weyl representation.
3. Write the Dirac equation in the Weyl representation and in the notation $\Psi_D = (\phi, \chi)^T$. Take the limit $m = 0$, and check if the components ϕ and χ satisfy the Klein-Gordon equation. Find the solution of this equation for the massless particle propagating along the x -direction.
4. Find the representation of the Dirac matrices α', β' such that $\text{Im } \alpha'_i = \text{Re } \beta' = 0$.

3. One useful relation

1. Show that

$$(\vec{\sigma} \cdot \vec{\pi})(\vec{\sigma} \cdot \vec{\pi}) = \vec{\pi}^2 - \frac{e\hbar}{c} \vec{\sigma} \cdot \vec{B}, \quad (6)$$

where $\vec{\pi} = \vec{p} - \frac{e}{c} \vec{A}$, $\vec{B} = \text{rot } \vec{A}$, and $\vec{\sigma}$ denotes the triplet of the Pauli matrices.

4. On Landau levels

In this exercise we are interested in energy levels of an electron in a uniform magnetic field. To find them, one should proceed in the same way as in the non-relativistic case. Namely, we take an *Ansatz* for stationary states,

$$\Psi = e^{-\frac{i}{\hbar} Et} \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad (7)$$

and plug it into the Dirac equation in the external field,

$$-\frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = \left(c \vec{\alpha} \cdot \left(-i\hbar \vec{\nabla} - \frac{e}{c} \vec{A} \right) + \beta mc^2 + e\Phi \right) \Psi. \quad (8)$$

This gives an eigenvalue problem for E whose solution will provide us with the desired energy levels.

Specifically, let us align the magnetic field along z -direction,

$$\vec{B} = \begin{pmatrix} 0 \\ 0 \\ \mathcal{B} \end{pmatrix}. \quad (9)$$

We will work in the Dirac representation of the matrices α_i, β studied in Lectures. For simplicity, we also put $\hbar = c = 1$.

1. Show that the magnetic field (9) is reproduced by the following combination of the potentials,

$$\vec{A} = - \begin{pmatrix} y\mathcal{B} \\ 0 \\ 0 \end{pmatrix}, \quad \Phi = 0. \quad (10)$$

Is this choice of $\vec{A}$ and Φ unique?

2. Substituting the Ansatz (7) and the potentials (10) into eq. (8), obtain an equation on the component ϕ .
3. Next, assume the following form of the general solution of the equation above,

$$\phi = e^{i(p_x x + p_z z)} \left(c_1 \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} \right), \quad (11)$$

with c_1, c_2, p_x, p_z arbitrary constants. Find equations on the functions $F_+(y)$ and $F_-(y)$. By changing variables, reduce them to the form

$$\left(\frac{d^2}{d\xi^2} - \xi^2 + \alpha_{\pm} \right) F_{\pm}(\xi) = 0. \quad (12)$$

4. Eq. (12) is of Hermite's type. It admits solutions provided that $\alpha_{\pm} = 2n + 1$, $n = 0, 1, 2, \dots$. Having this in mind, derive the formula for the electron energy levels. What is the degeneracy of the ground level $n = 0$? Of the first excited level $n = 1$?