Plasma I exercises for oral exam

Series	ex1	ex2	ex3
1	x	✓	✓
2	✓	✓	
3	x	✓	✓
4	x	✓	✓
5	✓	✓	
6	✓	✓	
7	✓	x	
8	x	✓	
9	x	✓	
10	x	x	
11	x	✓	
12	✓	✓	
13	x	x	
14	x		

Series 1, Exercise 2

To produce a plasma, a cylindrical vacuum tube (length $l = 2$m, radius $a = 1$m) is pumped down to a base pressure $p < 10^{-8}$torr (760torr is the atmospheric pressure), and then filled with Argon (ionization energy 15.8 eV) at $p = 10^{-3}$torr that you can assume to be at room temperature. A mono-energetic electron beam (radius $r = 10$cm, total current $I = 1$A and energy $U_e = 30$eV) is injected along the axis of the tube using an electron gun.

a.) Evaluate the total number of ions per second produced in the tube by the impact of the electrons on the neutrals.

b.) Assuming no recombination and no fueling of gas, how long could we maintain this discharge?

Suppose to have an ionization cross-section for electrons at 30eV of $\sigma_{ion} \sim 10^{-20}\text{m}^2$.

Recall:
1 Torr = 133 Pa

Series 1, Exercise 3

Semi-conductor manufacturers use plasma during surface treatment of materials. In a vacuum chamber of 0.5m $\times$ 0.5m $\times$ 0.5m dimensions, an inert gas is partially ionized by radio waves.

Consider the case where the gas used is Argon that you can assume to be at room temperature ($p = 10^{-4}$torr , $n_e = 10^{16}\text{m}^{-3}$, $T_e = 3\text{eV}$, and $T_i = 0.1\text{eV}$ - first ionization):

a.) Calculate the relative ionization degree of the gas used.

b.) Estimate the collision frequency (electron-neutral ν_{en}) assuming a cross section of $\sigma = 1000\pi a_0^2$, where $a_0 = 5.29 \times 10^{-11}$m is the Bohr radius.

c.) Can we consider this gas as a plasma? Why?

Recall:
1 Torr = 133 Pa

Series2, Exercise 1

A partially ionized slab of plasma has the following density distribution:

$$n(x) = n_0 cos\left(\frac{\pi x}{2L}\right), \qquad -L \leq x \leq L$$

Diffusion and recombination processes are present and a source term ensures $\partial n/\partial t = 0$. The recombination gives a reduction term equal to $-\alpha n^2$ in the continuity equation. Consider a constant diffusion term $D = 0.1\mathrm{m}^2/\mathrm{s}$, $\alpha = 10^{-15}\mathrm{m}^3/\mathrm{s}$ and $L = 2\mathrm{m}$.

a.) Find an equation expressing the global balance of the total number of plasma particles, taking into account the global losses due to diffusion and the ones due to recombination. Use this equation to find the rate at which particles must be injected by the source to maintain the given density distribution, that is express $\int_{-L}^{L} S(x)dx$ as a function of D, α, L and n_0.

b.) Find the value of the peak density (n_0) that would correspond to having the rate of losses at the wall equal to the recombination rate.

c.) Estimate the plasma particle mean life-time for that value of density.

Series 2, Exercise 2

Consider a weakly ionized hydrogen plasma in a long cylindrical vacuum tube (length L, radius r_c, and $L >> r_c$) with $T_e = T_i$. Two plates of radius $r_s < r_c$ are located at the two ends of the cylinder and biased to generate an RF field, which acts as a source of plasma. The plasma source can be assumed equal to a constant S_0 in the inner column of radius r_s and zero elsewhere. Measurements show that the plasma density drops to zero at the cylinder vessel, while the neutral density is constant. Since $L >> r_c$ one can neglect the axial losses at the end of the cylinder, and the plasma can be considered axially uniform. Moreover, recombination and turbulent phenomena can be neglected with respect to the dominant diffusion process.

a.) Write the diffusion coefficient describing the main transport process in such plasma.

b.) Find the steady state profile of the plasma density as a function of the radius, $n(r)$.

c.) Estimate the relative ionization degree (i.e. $n(r = 0)/n_n$) of the plasma at the center of the column.

Consider $T_e = T_i = 1$ eV, $\sigma_{i/n} \approx 10^{-18}$ m^2, $S_0 = 10^{19}$ m^{-3}s^{-1}, $r_s = 0.5$ m, $r_c = 1$ m, $L = 10$ m.

Series 3, Exercise 2

Consider the total momentum lost by a population of electrons colliding with a population of ions, in the three-dimensional space. Demonstrate that for a Maxwellian distribution of electrons with a drift velocity v_d ($v_d << v_{the}$) in the x direction, the average of the collision frequency is given by:

$$\bar{\nu}_p^{e/i} = \frac{1}{3}\sqrt{\frac{2}{\pi}}\nu_p^{e/i}(v_{the}) \simeq 0.26 \cdot \nu_p^{e/i}(v_{the})$$

where $\nu_p^{e/i}(v_{the})$ is the collision frequency for the momentum transfer between electrons and ions at the electron velocity v_{the}.

Indications:

- consider the physical meaning of the effective collision frequency to determine which is the physical quantity that has to be averaged.
- suppose $\ln \Lambda = const$, independent of the velocity and equal for electrons and ions.

Recall:

$$\nu_p^{e/i}(v_e) = n_i \frac{Z^2 e^4}{4\pi\epsilon_0^2} \frac{ln\Lambda}{m_e^2 v_e^3}$$

Series 3, Exercise 3

Consider the relaxation process of *alpha* particles (α's) at 3.5 MeV created by fusion reactions in a deuterium-tritium plasma (50 : 50 D-T). Evaluate the time-scale for the energy loss of α's in a plasma with $n_e = 10^{20}\,\mathrm{m}^{-3}$. Consider the collisions between three plasma species, assuming $T_e = T_D = T_T = 10$ keV.

a.) Which species is the most important in the α's thermalisation process?

b.) Which species is heated more by α's particles?

Suggestion: start with a thermal energy for the α's of 3.5 MeV and then consider the different regimes corresponding to the different energies of the α's during the thermalisation.

Recall:

$$\nu_{E_k}^{j/k} \sim n_k \frac{Z_j^2 Z_k^2 e^4}{2\pi\epsilon_0^2} \frac{ln\Lambda_k}{m_j m_k v_j^3}$$

Series 4, Exercise 2

Consider the electrons in the *"tail"* of the distribution function ($v >> v_{the}$). Show that the collision frequency for these electrons is:

$$\nu = \nu_p^{e/e'} + \nu_p^{e/i} = (2+Z)\frac{n_e e^4}{4\pi\epsilon_0^2}\frac{\ln\Lambda}{m_e^2 v^3}$$

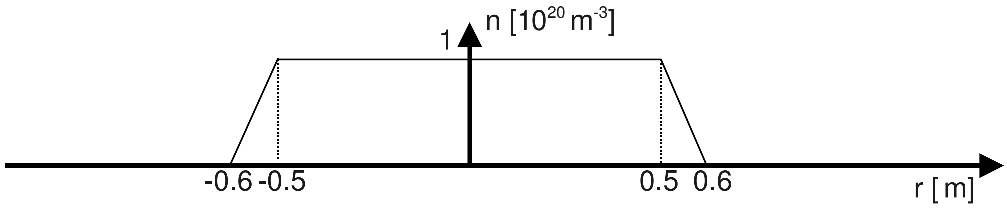
Demonstrate that these energetic electrons can be continuously accelerated (*run-away* regime) if their energy is higher than a critical value corresponding to a critical electric field, the Dreicer electric field, E_D:

$$\frac{1}{2}m_e v^2 > T_e\frac{E_D}{E}$$

Find an expression for E_D ~~and estimate the critical kinetic energy needed for an electron to enter the run-away regime for the electric field found in Exercise 1 using the same parameters~~.

Series 4, Exercise 3

Consider a fully ionized hydrogen plasma in a tokamak (major radius R, minor radius a) with a toroidal magnetic field B which is considered to be constant in the plasma (reasonable if $a \ll R$). The particle and heat sources are provided by injection of both neutral beam and electromagnetic waves. The steady state density profile in the radial direction is experimentally measured and is shown in the figure below:



Consider $B = 2\,\mathrm{T}$, $T_e = T_i = 10\,\mathrm{keV}$, $n(r = 0) = 10^{20}\,\mathrm{m}^{-3}$, $R = 2\,\mathrm{m}$, $a = 0.6\,\mathrm{m}$.

a.) Since a source is needed to maintain such steady state, it is obvious that the plasma is flowing out radially despite the magnetic confinement. What process could explain particle transport across the magnetic field? Is it possible to assert that the measured density profile is compatible with a diffusive particle transport?

b.) The measured particle flux between $r = 0.5\,\mathrm{m}$ and $r = 0.6\,\mathrm{m}$ is $\Gamma_n = 8 \times 10^{20}\,\mathrm{m}^{-2}\mathrm{s}^{-1}$. Calculate the effective diffusion coefficient D_{eff} at this location, and compare it with the classical diffusion coefficient $D_\perp$ that you would get considering the main collisional process.

c.) What can you conclude concerning the particle transport mechanism in this plasma?

Series 5, Exercise 1

a.) Consider the magnetic flux through a given surface moving with the plasma,

$$\Phi_B(t) \equiv \int\int_{S(t)} \mathbf{B}(t) \cdot \mathbf{dS}$$

Starting from the ideal MHD equations, show that the magnetic flux is *frozen* in the plasma,

$$\frac{d\Phi_B}{dt} = 0$$

What does this imply on the magnetic topology in such a plasma?

b.) Consider now that the plasma has some finite resistivity. How does this affect the magnetic flux?

c.) Find the diffusion equation for the magnetic field in a resistive plasma. Estimate the diffusion time of the magnetic field in ITER (characteristic length $L = 3\,\mathrm{m}$, electron temperature $T_e = 10\,\mathrm{keV}$).

Recall, the relevant ideal MHD equations are:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \mathbf{E} + \mathbf{u} \times \mathbf{B} &= 0 \end{aligned}$$

Series 5, Exercise 2

a.) ~~Demonstrate that the propagation of a transverse wave along the z axis ($\boldsymbol{k} = k\boldsymbol{e_z}$) in a string with tension S and mass per unit length M is given by:~~

$$\frac{\partial^2 y}{\partial z^2} = \frac{M}{S}\frac{\partial^2 y}{\partial t^2}$$

b.) Considering the ideal MHD model, demonstrate that the *Alfvén* waves (or *shear* waves), propagating along the magnetic field ($\boldsymbol{k} \parallel \boldsymbol{B_0}$, $\boldsymbol{B_0} = B_0\boldsymbol{e}_z$), can be described with the same equation of a transverse wave in a string. Identify the terms M and S in the equation in a.) for the Alfvén waves.

c.) The tokamak ITER will operate with a plasma D - T at 13 keV with a uniform electron density $n_e = 10^{20}\,\mathrm{m}^{-3}$ and a magnetic field $B = 6\,\mathrm{T}$. Evaluate the phase velocity of the Alfvén waves for that plasma.

d.) Fusion reactions $D^+ + T^+ \to He^{++}_{(3.5\mathrm{MeV})} + n_{(14\mathrm{MeV})}$ occur when a plasma is heated with ion beams D^+ with energy of 1 MeV. Which charged particles are resonant with the Alfvén waves (same velocity with wave)?

Remark: skip part a)

Recall, the linearised ideal MHD equations are:

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}_1 = 0 \\
&\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla p_1 + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 \\
&\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \\
&p_1 = c_s^2 \rho_1
\end{aligned}$$

Series 6, Exercise 1

In the lecture we have neglected the effect of collisions in the two-fluid model used to derive the dispersion relation of a wave in a magnetized plasma.

Consider a cold unmagnetized fluid plasma ($T = 0, \boldsymbol{B_0} = 0$).

a.) Derive the dispersion relation of waves in such plasma keeping the collision term in the momentum equation for the electrons.

b.) Show that in this case longitudinal waves (*Langmuir* waves) are damped.

Recall, the wave equation for $\mathbf{E}$ is:

$$\left[N^2 \left(\frac{\mathbf{kk}}{k^2} - \mathbf{1} \right) + \overline{\epsilon} \right] \cdot \mathbf{E} = 0$$

$$\overline{\epsilon} = \mathbf{1} \left(1 + \frac{i}{\epsilon_0 \omega} \, \bar{\sigma} \right)$$

Series 6, Exercise 2

An antenna can detect frequencies around $f = 80\,\mathrm{MHz}$ and is used to measure the wave coming from a pulsar producing a broad electromagnetic spectrum.

Due to the dispersion of the group velocity caused by the interstellar plasma, the measured frequency during a pulse drift varies according to $\mathrm{d}f/\mathrm{d}t = -5\,\mathrm{MHz}\cdot\mathrm{s}^{-1}$.

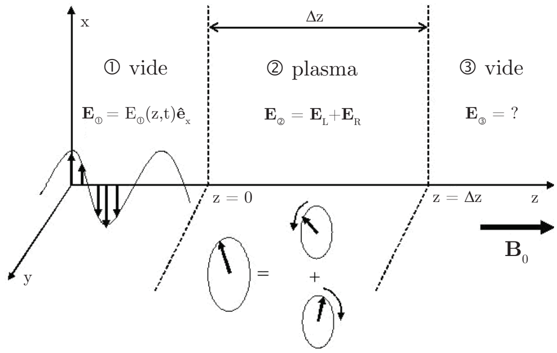
a.) Considering $\omega^2 \gg \omega_p^2$ and neglecting the magnetic field in the interstellar plasma, demonstrate that:

$$\frac{\mathrm{d}f}{\mathrm{d}t} \approx -\frac{c}{x}\frac{f^3}{f_p^2}$$

where $f_p = \omega_p/2\pi$ and x is the distance of the pulsar.

b.) Find the distance of the pulsar in *parsec* ($1\,\mathrm{parsec} = 3\times10^{16}\,\mathrm{m}$) considering a mean electron density in space of $2\times10^{6}\,\mathrm{m}^{-3}$.

Series 7, Exercise 1



Consider the situation displayed above. An electromagnetic wave propagating along the magnetic field B_0 crosses a portion of plasma. In the plasma, consider the dispersion relation of an electromagnetic wave in a cold uniform plasma :

$$N^2 = \frac{k^2c^2}{\omega^2} = \frac{(\omega \mp \omega_R)(\omega \pm \omega_L)}{(\omega \pm \Omega_i)(\omega \mp |\Omega_e|)} \approx 1 - \frac{\omega_{pe}^2/\omega^2}{1 \mp |\Omega_e|/\omega}; \quad (\omega \gg \Omega_i)$$

The upper sign is related to the right-handed wave (R) and the lower sign to the left-handed wave (L).

a.) Show that the rotation (polarization) angle α, when the wave exits the plasma, is equal to half of the phase difference between the two waves. Find a relation for α as a function of the distance travelled, ω, Ω_e and ω_{pe}. Consider the limit:

$$\frac{\omega_{pe}^2/\omega^2}{1 \mp |\Omega_e|/\omega} \ll 1$$

b.) The Faraday rotation of a micro-wave beam ($\lambda = 8\,\text{mm}$) in a uniform plasma with a magnetic field $B_0 = 0.1\,\text{T}$ is measured. When the beam propagates through 1 m of plasma, the polarization direction turns of 90°. Find the plasma density.

Series 8, Exercise 2

Show that in a plasma described by the *Vlasov* equation:

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{x}} + \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

the entropy, defined in a simple way (without numerical factors) as:

$$S(t) = -\int d\vec{v} \int d\vec{x} f(\vec{x}, \vec{v}, t) \ln\left(f(\vec{x}, \vec{v}, t)\right)$$

is conserved ($dS/dt = 0$).

Series 9, Exercise 2

Using the general dispersion relation from the *Vlasov-Maxwell* model:

$$D(\omega, k) = 1 + \sum_{\alpha} \frac{e^2}{m_\alpha \epsilon_0 k} \int_{-\infty}^{+\infty} du \frac{dF_{0\alpha}}{du} \frac{1}{\omega - ku} = 0$$

evaluate the dispersion relation of the ion-acoustic waves in the limit $kv_{thi} \ll \omega \ll kv_{the}$, $T_e \gg T_i$, and assuming $\lambda \sim 1/k \gg \lambda_D$. Consider F_{0e} and F_{0i} as Maxwellian distribution functions.

Notice that in the case of waves with low frequency (e.g. the ion-acoustic waves), both species have to be considered (electrons and ions).

Series 11, Exercise 2

Evaluate the **Landau damping** for an **ion-acoustic wave** solution of the dispersion relation of the *Vlasov-Poisson* model,

$$D(\omega, k) = \epsilon(\omega, k) = 1 - \sum_{\alpha} \frac{e^2}{m_\alpha \epsilon_0 k^2} \int_{\mathcal{L}} \mathrm{d}u \frac{\mathrm{d}F_{\alpha 0}}{\mathrm{d}u} \frac{1}{u - \frac{\omega}{k}} = 0.$$

where the integral should now be evaluated using Landau's rule. Suppose to have a maxwellian equilibrium distribution. Assuming that $kv_{thi} \ll \omega \ll kv_{the}$, $T_e \gg T_i$ and $\lambda \gg \lambda_D$, show that the total damping rate of the wave is $\gamma_t = \gamma_e + \gamma_i$, where γ_e and γ_i are respectively the electron and ion contributions,

$$\gamma_e \approx -\sqrt{\frac{\pi}{8}} k c_s \sqrt{\frac{m_e}{m_i}}$$

$$\gamma_i \approx -\sqrt{\frac{\pi}{8}} k c_s \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{T_e}{2T_i}\right)$$

Series 12, Exercise 1

Consider a quasi-neutral electron-proton plasma in which an equilibrium current is flowing. This may be described by a Maxwellian ion distribution at rest and a drifting Maxwellian for the electrons

$$F_i(u) = \frac{n}{\sqrt{2\pi} v_{th,i}} \exp\left[-\frac{u^2}{2v_{th,i}^2}\right] \qquad F_e(u) = \frac{n}{\sqrt{2\pi} v_{th,e}} \exp\left[-\frac{(u-v_d)^2}{2v_{th,e}^2}\right]$$

where $v_{th,i}, c_s \ll v_d \ll v_{th,e}$.

a.) Make a plot of the distribution functions of ions and electrons on the same scale, look in the region, $v_{th,i} < \omega_r/k \ll v_{th,e}$, and show where you expect unstable waves might occur.

b.) Consider an ion-acoustic wave: write an expression for the damping/growth rate, γ, including both electron and ion contributions. Show that the electron contribution introduces a destabilizing term in the expression of γ.

c.) Demonstrate that the condition $T_e \gg T_i$ is generally required for instability and justify the result. Show that $\gamma \sim \sqrt{\frac{\pi}{8}} k v_d (m_e/m_i)^{1/2}$ when $T_e \gg T_i$.

Recall:

$$D(\omega, k) = \epsilon(\omega, k) = 1 - \sum_\alpha \frac{e^2}{m_\alpha \epsilon_0 k^2} \int_{\mathcal{L}} \mathrm{d}u \frac{\mathrm{d}F_{\alpha 0}}{\mathrm{d}u} \frac{1}{u - \frac{\omega}{k}} = 0.$$

Series 12, Exercise 2

Consider a uniform plasma with a fixed population of ions and two different electron populations:

- a Maxwellian population with density n_p, temperature T_p and no drift velocity
- a Maxwellian beam with density n_b, temperature T_b and drift velocity $\mathbf{v} = V\mathbf{e_x}$

When the magnitude of the beam density n_b exceeds a certain threshold the two-stream instability can develop. As seen in the lecture, the Landau damping coefficient γ is proportional to the imaginary part of the dielectric function $\epsilon_i(\omega_r)$. Its sign determines wether a given mode can become unstable or not. Supposing that the phase velocity of the instability, v_ϕ, corresponds to a velocity v for which the slope of $f_b(v)$ is maximum and supposing that $V \gg v_{th,b}$, show that the critical density ratio above which there can be an instability is:

$$\frac{n_b}{n_p} = \sqrt{e}\frac{T_b}{T_p}\frac{V}{v_{th,p}}\exp\left(-\frac{V^2}{2v_{th,p}^2}\right).$$