7.15 NONLINEAR ION-ACOUSTIC WAVES—KORTEWEG–DeVRIES EQUATION

Up to this point in the fluid theory we have considered only *linear* waves. We must always remember that the theory of linear waves restricts us to very small amplitudes. A wave with a finite amplitude will be susceptible to nonlinear effects, which show up mathematically as products of first order terms. This section and the next section are intended to introduce the concept of nonlinear wave equations and their corresponding solutions, which often take the form of solitons and shock waves.

Here we consider an example of one nonlinear wave equation, the Korteweg–deVries equation [20]:

$$\partial_t v + v\,\partial_z v + \alpha \partial_z{}^3 v = 0 \tag{7.257}$$

This equation is obtained by adding one nonlinear term in the derivation of the ion-acoustic wave equation.

Although it is possible to give a rigorous derivation of (7.257), we give here only a heuristic derivation that indicates how one might arrive at (7.257). The origin of the terms in (7.257) is fairly easy to see. The first two terms might arise from the total time derivative of the ion fluid velocity. The third term can be seen in the ion-acoustic dispersion relation (7.104), which upon taking $T_i = 0$, $\gamma_e = 1$, is

$$\omega^2 = \frac{k^2 c_s{}^2}{1 + k^2 \lambda_e{}^2} \tag{7.258}$$

The square root of (7.258) is, for small $k\lambda_e$,

$$\omega = \frac{k c_s}{(1 + k^2 \lambda_e{}^2)^{1/2}} = k c_s \left(1 - \frac{k^2 \lambda_e{}^2}{2}\right) \tag{7.259}$$

If we now multiply (7.259) on the right by the ion fluid velocity v, and identify $-i\omega$ with ∂_t and ik with ∂_x, we obtain

$$\frac{\partial v}{\partial t} = -\,c_s \frac{\partial v}{\partial x} - \frac{c_s \lambda_e{}^2}{2} \frac{\partial^3 v}{\partial x^3} \tag{7.260}$$

In a frame $x' = x - c_s t$ moving with the velocity c_s, and defining $\alpha = c_s \lambda_e{}^2/2$, we obtain

$$\partial_t v + \alpha \partial_{x'}{}^3 v = 0 \tag{7.261}$$

which are the linear terms in (7.257). The nonlinear term is obtained by replacing the partial time derivative ∂_t with the convective time derivative $\partial_t + v\partial_{x'}$.

We begin our heuristic derivation with the five fluid equations. Taking $T_i \to 0$ so that we can neglect ion pressure in the ion force equation, and taking $m_e \to 0$ so that we can neglect electron inertia in the electron force equation, we find

$$\partial_t n_e + \partial_x (n_e V_e) = 0 \tag{7.262}$$

$$0 = -\,T_e\,\partial_x n_e - e n_e E \tag{7.263}$$

$$\partial_t n_i + \partial_x (n_i V_i) = 0 \tag{7.264}$$

$$m_i n_i\,\partial_t V_i + m_i n_i V_i\,\partial_x V_i = e n_i E \tag{7.265}$$

and

$$\partial_x E = 4\pi e (n_i - n_e) \tag{7.266}$$

where we have chosen $\gamma_e = 1$. We next linearize (7.262) to (7.266) everywhere except one place: we keep one nonlinear term, the $m_i n_0 V_i \partial_x V_i$ term on the left side of (7.265). We have then

$$\partial_t n_{e1} + n_0 \partial_x V_e = 0 \tag{7.267}$$

$$0 = -T_e \partial_x n_{e1} - en_0 E \tag{7.268}$$

$$\partial_t n_{i1} + n_0 \partial_x V_i = 0 \tag{7.269}$$

$$m_i n_0 \partial_t V_i + m_i n_0 V_i \partial_x V_i = en_0 E \tag{7.270}$$

$$\partial_x E = 4\pi e(n_{i1} - n_{e1}) \tag{7.271}$$

EXERCISE Can you find seven other nonlinear terms neglected in going from (7.262)–(7.266) to (7.267)–(7.271)?

A more rigorous derivation would show us the regime of validity implied by our neglect of seven other nonlinear terms while retaining only one nonlinear term. It turns out that this regime is reasonably large.

We next assume a plane wave solution, everywhere *except in (7.270)*. [What would happen if we tried to assume a plane wave solution $\sim \exp(-i\omega t + ikx)$ in (7.270)?] We also take $v \equiv V_e \approx V_i$, which means that (7.267) and (7.269) have the same information; we retain the difference between n_{e1} and n_{i1} so that (7.271) can be used. Solving (7.268) for n_{e1}, we find

$$n_{e1} = -\frac{en_0 E}{ikT_e} \tag{7.272}$$

which inserted in Poisson's equation (7.271) yields

$$E = \frac{4\pi e n_{i1}}{ik - (\omega_i^2 m_i / ikT_e)} \tag{7.273}$$

n_{i1} is from (7.269)

$$n_{i1} = \frac{kn_0}{\omega} v \tag{7.274}$$

Both (7.274) in (7.273) and the result in (7.270) yield

$$\partial_t v + v\partial_x v = -\frac{ik^2 c_s^2}{\omega}(1 + k^2\lambda_e^2)^{-1} v \tag{7.275}$$

Here, we are still treating the right side as linear; therefore ω and k have their meanings as differential operators, while the left side is nonlinear. It proves convenient to eliminate ω on the right side; we do this by using the linear ion-acoustic dispersion relation (7.258), which is obtained from (7.275) by ignoring the nonlinear term and replacing the left side with $-i\omega v$. Solving for ω and substituting in the right side of (7.275), we have

$$\partial_t v + v\,\partial_x v = -ikc_s(1 + k^2\lambda_e^2)^{-1/2} v \tag{7.276}$$

For small $k\lambda_e$, we can expand the right side of (7.276) to obtain

$$\partial_t v + v\,\partial_x v = -ikc_s\left(1 - \frac{1}{2}k^2\lambda_e^2\right) v \tag{7.277}$$

Reinterpreting ik as ∂_x, this becomes

$$\boxed{\partial_t v + (c_s + v)\,\partial_x v + \alpha\,\partial_x^3 v = 0} \tag{7.278}$$

where $\alpha = \lambda_e^2 c_s/2$. In the frame $z = x - c_s t$, this is the Korteweg–deVries equation (7.257).

EXERCISE Show the above relationship.

Recall that $v(x,t)$ represents fluid velocity in the laboratory frame; this identification of $v(x,t)$ remains true even if we transform to a moving frame. What physics do the various terms in (7.278) represent? The first two terms by themselves,

$$\partial_t v + c_s\,\partial_x v = 0 \tag{7.279}$$

merely represent our old ion-acoustic waves in the limit $k\lambda_e \to 0$. The solution of (7.279) is simply a dispersionless wave, $\omega = kc_s$, with phase velocity $V_\varphi \equiv \omega/k = c_s$, and group velocity $d\omega/dk = c_s$ a constant independent of k. Suppose we add the nonlinear term to obtain

$$\partial_t v + (c_s + v)\,\partial_x v = 0 \tag{7.280}$$

The effect of the nonlinear term is as follows. Consider an initial waveform as shown in Fig. 7.26. As the wave moves, the part with larger v moves faster, so that it overtakes the part with smaller v. Eventually, at $t = t_2$, there is an infinite slope, and at $t = t_3$, the wave has broken. Now suppose we had included the dispersive term in (7.278); the term $\alpha\,\partial_x^3 v$ is called dispersive because it contributes a term k^3 to the linear dispersion relation $\omega = kc_s - \alpha k^3$; then $V_g \equiv d\omega/dk = c_s - 3\alpha k^2$, which depends on k, making this a dispersive wave. We know the effect of dispersion on a wave; it makes a wave packet spread out as it travels. This is just opposite to the steepening observed in the figure. Consider the time between $t = t_1$ and $t = t_2$. Here, the slope is becoming very large. A large slope corresponds to a large x-derivative, which makes the $\alpha\,\partial_x^3 v$ term in (7.278) become large. Since we know that the effect of this $\alpha\,\partial_x^3 v$ term is to spread out the wave, we might expect that there could be a balance between the nonlinear steepening and the linear dispersion. Indeed this is the case. One can obtain nonlinear wave packets, known as ***solitons***, which travel without change of shape (Fig. 7.27). The physical basis for these solitons involves a balance between dispersion and nonlinearity.

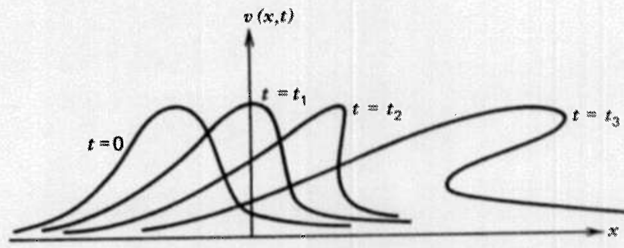

Fig. 7.26 Effect of the nonlinear term in (7.282).

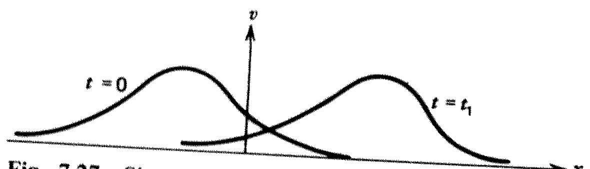

Fig. 7.27 Sketch of a soliton solution.

Let us proceed to find a soliton solution to the Korteweg–deVries equation (7.278). We look for stationary solutions in a moving frame,

$$x' = x - v_0 t \tag{7.281}$$

$$t' = t \tag{7.282}$$

so that

$$\partial_x = (\partial x'/\partial x)\partial_{x'} + (\partial t'/\partial x)\partial_{t'} = \partial_{x'} \tag{7.283}$$

and

$$\partial_t = (\partial x'/\partial t)\partial_{x'} + (\partial t'/\partial t)\partial_{t'} = \partial_{t'} - v_0\partial_{x'} \tag{7.284}$$

Since stationary implies $\partial_{t'} = 0$, the Korteweg–deVries equation (7.278) becomes

$$(-v_0 + c_s + v)\,\partial_{x'}v + \alpha\,\partial_{x'}{}^3 v = 0 \tag{7.285}$$

Remember that $v(x',t')$ is still that function of space and time which represents the fluid velocity in the lab frame. Equation (7.285) can be integrated once immediately, to give

$$(c_s - v_0)v + \frac{v^2}{2} + \alpha v'' = 0 \tag{7.286}$$

where $(\ )' \equiv \partial_{x'}(\ )$ and we have taken the integration constant to vanish. Equation (7.286) is in the form

$$\alpha v'' = (v_0 - c_s)v - \frac{v^2}{2} \tag{7.287}$$

which has the same mathematical form as Newton's law of motion,

$$m\ddot{x} = F(x) = -\partial_x V(x) \tag{7.288}$$

where $V(x)$ is the potential energy. Thus, (7.287) has the form

$$\alpha v'' = -\partial_v\left[(c_s - v_0)\frac{v^2}{2} + \frac{v^3}{6}\right] \tag{7.289}$$

Equation (7.289) has the same mathematical form as a force equation for a particle of mass α moving under the influence of a potential field given by the quantity in brackets. We call the quantity in brackets the pseudopotential,

$$\Phi(v) = (c_s - v_0)\frac{v^2}{2} + \frac{v^3}{6} \tag{7.290}$$

A graph of $\phi(v)$ is shown for $c_s - v_0 > 0$ in Fig. 7.28. A similar graph of the pseudopotential $\Phi(v)$ for $(c_s - v_0) < 0$ is shown in Fig. 7.29. Only the second form is suitable for our purposes. This is because we desire a localized wave form, $v(x' \to \pm\infty) \to 0$. This will occur in Fig. 7.29 when the pseudoparticle leaves $v = 0$ when the pseudotime $x' \to -\infty$, falling once through the well to reach v_{max}

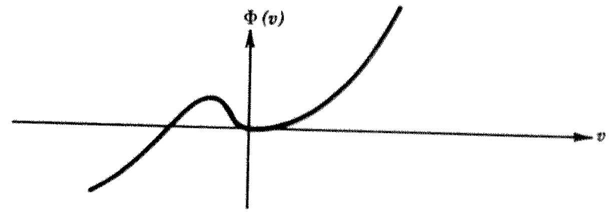

Fig. 7.28 Sketch of pseudopotential when $c_s > v_0$.

at $x' = 0$, and taking an infinite amount of pseudotime x' to fall back through t well to reach $v = 0$ as the pseudotime $x' \to +\infty$. We thus obtain the shape show in Fig. 7.30. The pseudopotential in Fig. 7.28 would not allow $v(x' \to \pm\infty) \to$

Let us now solve (7.287) exactly, with $c_s - v_0 < 0$ or $v_0 > c_s$. We all kno how to solve force equations of the form (7.287). Multiply (7.287) by v' an integrate, to obtain,

$$\frac{\alpha}{2}(v')^2 = (v_0 - c_s)\frac{v^2}{2} - \frac{v^3}{6} \tag{7.29}$$

where we have chosen the constant of integration to be zero because we war $v' = 0$ when $v = 0$ (Fig. 7.30). Then

$$\frac{dv}{dx'} = v' = \left(\frac{2}{\alpha}\right)^{1/2}\left[(v_0 - c_s)\frac{v^2}{2} - \frac{v^3}{6}\right]^{1/2} \tag{7.292}$$

or

$$\frac{dv}{\left[\frac{(v_0 - c_s)}{2}v^2 - \frac{v^3}{6}\right]^{1/2}} = \left(\frac{2}{\alpha}\right)^{1/2} dx' \tag{7.293}$$

Each side of (7.293) can be integrated. The left side is of the form

$$I = \int \frac{dv}{\sqrt{v^2 - \beta v^3}} = \int \frac{dv}{v\sqrt{1 - \beta v}} \tag{7.294}$$

where $\beta = 1/[3(v_0 - c_s)]$. Let $u = \sqrt{1 - \beta v}$, then $v = (1 - u^2)/\beta$ and $du =$ $(-\beta\, dv/2)/\sqrt{1 - \beta v}$. We find

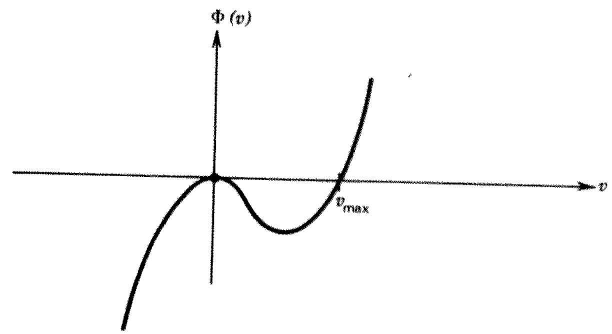

Fig. 7.29 Sketch of pseudopotential when $c_s < v_0$.

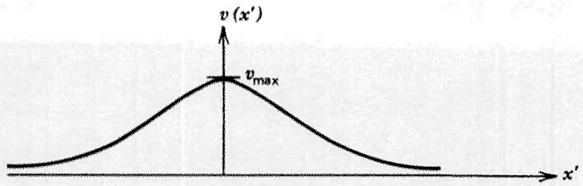

Fig. 7.30 Sketch of soliton solution to the Korteweg–deVries equation.

$$I = -2\int \frac{du}{1-u^2} = -\int du \left(\frac{1}{1-u} + \frac{1}{1+u}\right) = \ln\left(\frac{1-u}{1+u}\right) \tag{7.295}$$

Then

$$\left(\frac{2}{v_0 - c_s}\right)^{1/2} \ln\left(\frac{1-u}{1+u}\right) = \left(\frac{2}{\alpha}\right)^{1/2} x' \tag{7.296}$$

from (7.293). With $\gamma \equiv [(v_0 - c_s)/\alpha]^{1/2}$, and exponentiating both sides, we get

$$\frac{1-u}{1+u} = e^{\gamma x'} \tag{7.297}$$

Then $1 - u = (1 + u)e^{\gamma x'}$, implying that

$$u = \frac{1 - e^{\gamma x'}}{1 + e^{\gamma x'}} \tag{7.298}$$

and $v = (1 - u^2)/\beta$ is

$$v = \frac{1}{\beta}\left[\frac{(1 + e^{\gamma x'})^2 - (1 - e^{\gamma x'})^2}{(1 + e^{\gamma x'})^2}\right] = \frac{1}{\beta}\left[\frac{4e^{\gamma x'}}{(1 + e^{\gamma x'})^2}\right] \tag{7.299}$$

or

$$v = \frac{1}{\beta}\,\frac{4}{(e^{\gamma x'/2} + e^{-\gamma x'/2})^2} = \frac{1}{\beta}\,\mathrm{sech}^2(\gamma x'/2) \tag{7.300}$$

which is

$$\boxed{v = 3(v_0 - c_s)\,\mathrm{sech}^2\left[\left(\frac{v_0 - c_s}{4\alpha}\right)^{1/2} x'\right]} \tag{7.301}$$

In fact, this solution has only been derived for $x' < 0$ since we chose the $v' > 0$ branch in (7.292); nevertheless, it would be easy to obtain the part of (7.301) for $x' > 0$ by choosing the $v' < 0$ branch in (7.292); therefore, (7.301) applies to all x' and is the soliton solution. Note that the larger amplitude solitons are more sharply peaked, having a smaller scale length. This behavior is in accordance with our picture of nonlinearity $v\,\partial_x v$ which balances dispersion $\partial_x^{\,3} v$ (Fig. 7.31). Back in the lab frame, where $x = x' + v_0 t$, this solution is

$$\boxed{v(x,t) = 3(v_0 - c_s)\,\mathrm{sech}^2\left[\left(\frac{v_0 - c_s}{4\alpha}\right)^{1/2} (x - v_0 t)\right]} \tag{7.302}$$