Plasma Physics I

Solution to the Series 7 (November 2, 2024)

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Exercise 1

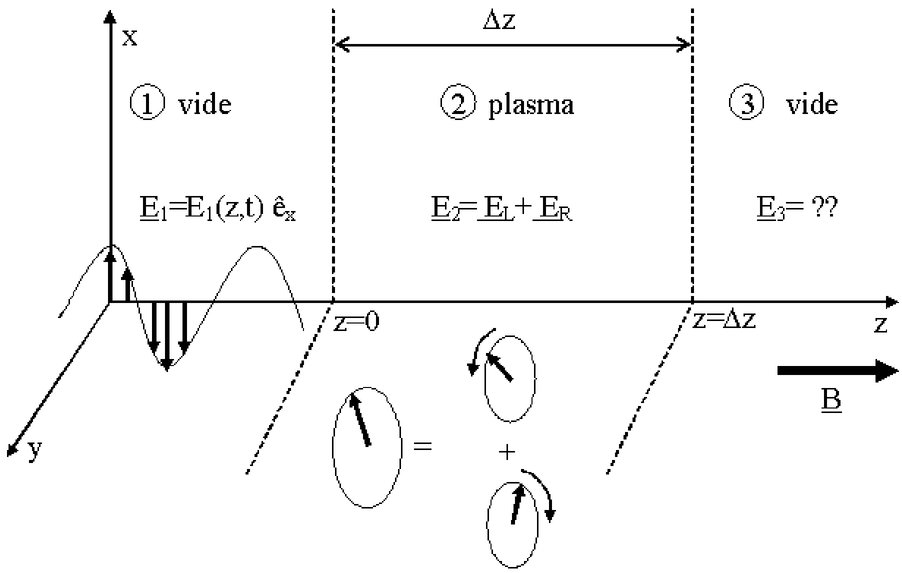

Figure 1: A wave that propagates along z on three different regions: vacuum - plasma - vacuum.

Region ①

In vacuum, we have:

$$\vec{E}(z,t) = \vec{E}_{①}(z,t) = E_0 \hat{e}_x \, e^{i(k_z\, z - \omega t)} \tag{1}$$

Notice that a linearly polarized wave can be seen as a combination of two circularly polarized waves with the same phase velocity.

Region ②

Due to the anisotropy induced by the magnetic field $\vec{B}_0$, the two modes R and L propagate through the plasma with different phase velocity. Indeed, the electric field of the R mode rotates in the same direction of the electron cyclotron motion while the L mode rotates in the opposite sense, that is that of the ions.

The electric field $\vec{E}$ is given by:

$$\vec{E} = \vec{E}_L + \vec{E}_R \tag{2}$$

therefore

$$\vec{E}_2(z,t) = \Big[E_L(\hat{e}_x - i\hat{e}_y)\, e^{ik_L\, z} + E_R(\hat{e}_x + i\hat{e}_y)\, e^{ik_R\, z}\Big]\, e^{i\omega t}, \tag{3}$$

where the wave vectors are:

$$k^2_{R,L} \approx \frac{\omega^2}{c^2}\left(1 - \frac{\omega^2_{\mathrm{pe}}/\omega^2}{1\mp \mid \Omega_e \mid /\omega}\right) \tag{4}$$

To evaluate the amplitudes E_R, E_L of the two modes, we can use the matching condition (for each t) at $z = 0$:

$$\vec{E}_{①}(z=0,t) = \vec{E}_{②}(z=0,t), \qquad \forall\ t \tag{5}$$

obtaining:

$$E_0\hat{e}_x = E_L(\hat{e}s_x - i\hat{e}_y) + E_R(\hat{e}_x + i\hat{e}_y) \tag{6}$$

and, using the condition $vecE = E\hat{e}_x$ for $z < 0$ (i.e. $E_y = 0$):

$$E_L = E_R = \frac{E_0}{2} \tag{7}$$

Region ③

Without plasma, the two modes propagate again with the same phase velocity. The wave is linearly polarized but the polarization angle is different according to the direction of the two modes at $z = \Delta z$.

The electric field $\vec{E}_{②}$ at the position $z = \Delta z$ is:

$$\vec{E}_{②}(z=\Delta z,t) = \frac{E_0}{2}\Big[(\hat{e}_x - i\hat{e}_y)\, e^{ik_L\,\Delta z} + (\hat{e}_x + i\hat{e}_y)\, e^{ik_R\,\Delta z}\Big]\, e^{-i\omega t} \tag{8}$$

a) *Evaluation of the rotation angle*

To find the direction of the wave leaving the plasma region ($\vec{n} = \vec{n}(\hat{e}_x, \hat{e}_y)$) we need to rewrite the eq.(8) as a function of the rotation angle α of the electric field around the z axis.

Using the identity:

$$k_L \equiv \frac{k_R + k_L}{2} + \frac{k_L - k_R}{2} \tag{9}$$

$$k_R \equiv \frac{k_R + k_L}{2} - \frac{k_L - k_R}{2} \tag{10}$$

we find:

$$\vec{E}_{②}(z=\Delta z,t) = \frac{E_0}{2}\, e^{i\left(\frac{k_R+k_L}{2}\Delta z - \omega t\right)}\Bigg[\hat{e}_x\left(e^{i\,\frac{k_L-k_R}{2}\Delta z} + e^{-i\,\frac{k_L-k_R}{2}\Delta z}\right) + \\ -i\hat{e}_y\left(e^{i\,\frac{k_L-k_R}{2}\Delta z} - e^{-i\,\frac{k_L-k_R}{2}\Delta z}\right)\Bigg] \tag{11}$$

and, from the relation:

$$e^{i\alpha} \equiv \cos\alpha + i\sin\alpha. \tag{12}$$

Considering the symmetry/asymmetry of the trigonometric functions, we find:

$$\vec{E}_{②}(z=\Delta z, t) = E_0\, e^{i\left(\frac{k_R+k_L}{2}\Delta z - \omega t\right)} \underbrace{\left[\hat{e}_x \cos\left(\frac{k_L-k_R}{2}\Delta z\right) + \hat{e}_y \sin\left(\frac{k_L-k_R}{2}\Delta z\right)\right]}_{\vec{\hat{n}}(z=\Delta z)\equiv\vec{\hat{n}}(z>\Delta z)} \tag{13}$$

The polarization angle α is then equal to:

$$\alpha = \frac{k_L - k_R}{2}\,\Delta z \tag{14}$$

If the condition

$$\frac{\omega_{\text{pe}}^2/\omega^2}{1\mp \mid \Omega_e \mid /\omega} \ll 1 \tag{15}$$

is verified, we can use the approximation $(1-x)^{1/2} \approx 1 - x/2$ and write k_R, k_L as:

$$k_{R,L} \approx \frac{\omega}{c}\left[1 - \frac{\omega_{\text{pe}}^2}{2\omega(\omega\mp \mid \Omega_e \mid)}\right]. \tag{16}$$

Finally[1]

$$\alpha \approx \frac{\Delta z}{2c}\,\frac{\omega_{\text{pe}}^2\ \mid \Omega_e \mid}{\omega^2 - \Omega_e^2} \tag{18}$$

[1]Dimensional analysis. The angle is given in $[rad]$ (i.e. without dimension):

$$[\alpha] = \frac{\text{m}}{\text{m}\cdot\text{s}^{-1}}\,\frac{\text{s}^{-2}\cdot\text{s}^{-1}}{\text{s}^{-2}} \to \text{O.K.} \tag{17}$$

b) *Density measurement*

In the region ② we have a uniform plasma with a magnetic field $B = 0.1T$. The wavelength of the microwave beam is $\lambda = 8mm$, that corresponds to a frequency:

$$\omega = \frac{2\pi c}{\lambda} = 2.36 \times 10^{11} \, \frac{\text{rad}}{\text{s}}. \tag{19}$$

The phase difference at $z = \Delta z$ is $\alpha = \pi/2$. Using the definition of plasma frequency and electron gyrofrequency:

$$\omega_{\text{pe}}^2 = \frac{e^2 \, n_e}{m_e \, \varepsilon_0}\,, \qquad\qquad \mid \Omega_e \mid = \frac{eB}{m_e} \tag{20}$$

in the eq.(18) we have

$$n_e \approx \frac{2\alpha \, m_e \, c \, \varepsilon_0}{\Delta z \, \mid \Omega_e \mid e^2} \, (\omega^2 - \Omega_e^2) \qquad\qquad \Rightarrow n_e \approx 9.32 \times 10^{17} \, \text{m}^{-3} \tag{21}$$

We still need to verify the assumption in the eq.(15). From the value of the density found above, we have:

$$\frac{\omega_{\text{pe}}^2/\omega^2}{1 \mp \mid \Omega_e \mid /\omega} \approx 0.05 \div 0.06 \ll 1. \tag{22}$$

Therefore we can conclude that the approximation is valid and the result is correct.

Exercise 2

Vlasov equation for the α species:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0$$

with $\mathbf{a} = \frac{q}{m} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$ or, in a different form:

$$\frac{\partial f_\alpha}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbf{v} f_\alpha) + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a} f_\alpha) = 0$$

There are two contributions to the total energy: the first, E_p, is coming from the particles and the other, E_f, is related to the electromagnetic field:

$$E_p(t) = \sum_\alpha \int d^3x \int d^3v \, \frac{1}{2} m_\alpha v^2 f_\alpha(\mathbf{x}, \mathbf{v}, t) \tag{23}$$

$$E_f(t) = \int d^3x \left(\frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) \tag{24}$$

For the particles:

$$
\begin{aligned}
\frac{dE_p}{dt} &= \sum_\alpha \int d^3x \int d^3v\, \frac{1}{2} m_\alpha v^2 \frac{\partial f_\alpha}{\partial t} \\
&= -\sum_\alpha \int d^3x \int d^3v\, \frac{1}{2} m_\alpha v^2 \left[\frac{\partial}{\partial \mathbf{x}} \cdot (\mathbf{v} f_\alpha) + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a} f_\alpha) \right] \\
&= -\sum_\alpha \int d^3v\, \frac{1}{2} m_\alpha v^2 \underbrace{\int d^3x \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbf{v} f_\alpha)}_{\int_S \mathbf{v} f \cdot d\vec{\sigma} = 0} - \sum_\alpha \int d^3x \int d^3v\, \frac{1}{2} m_\alpha v^2 \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a} f_\alpha)
\end{aligned}
\tag{25}
$$

Notice:

From Gauss's theorem:

$$
\int_V d^3x \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbf{v} f_\alpha) = \int_S \mathbf{v} f \cdot d\vec{\sigma}
$$

and this integral vanishes because as $|\mathbf{x}| \to \infty$, f approaches zero quickly.

$$
\frac{\partial}{\partial \mathbf{v}} \cdot (v^2 \mathbf{a} f_\alpha) = v^2 \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a} f_\alpha) + \mathbf{a} f_\alpha \cdot \frac{\partial}{\partial \mathbf{v}} \left(v^2\right) = v^2 \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a} f_\alpha) + \mathbf{a} f_\alpha \cdot 2\mathbf{v}
$$

$$
\frac{\partial}{\partial \mathbf{v}} \left(v^2\right) = \begin{pmatrix} \dfrac{\partial}{\partial v_x} \\[2ex] \dfrac{\partial}{\partial v_y} \\[2ex] \dfrac{\partial}{\partial v_z} \end{pmatrix} \left(v_x^2 + v_y^2 + v_z^2\right) = \begin{pmatrix} 2\, v_x \\ 2\, v_y \\ 2\, v_z \end{pmatrix} = 2\mathbf{v}.
$$

Therefore:

$$
\begin{aligned}
\frac{dE_p}{dt} &= -\sum_\alpha \int d^3x \int d^3v \left[\frac{1}{2} m_\alpha \underbrace{\frac{\partial}{\partial \mathbf{v}} \cdot (v^2 \mathbf{a} f_\alpha)}_{=0 \text{ using Gauss's theorem}} - q_\alpha (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_\alpha \cdot \mathbf{v} \right] \\
&= +\sum_\alpha \int d^3x \int d^3v\, f_\alpha q_\alpha \mathbf{v} \cdot \mathbf{E} = \int d^3x \underbrace{\int d^3v \sum_\alpha q_\alpha f_\alpha \mathbf{v}}_{\mathbf{j}} \cdot \mathbf{E} = \int d^3x\, \mathbf{j} \cdot \mathbf{E}
\end{aligned}
\tag{26}
$$

For the fields:

$$
\frac{dE_f}{dt} = \varepsilon_0 \int d^3x\, \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \int d^3x\, \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}
$$

$$\begin{cases} \dfrac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B} - c^2 \mu_0 \mathbf{j} \\ \\ \dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \end{cases}$$

$$\begin{aligned} \frac{dE_f}{dt} =& \varepsilon_0 \int d^3x \, \mathbf{E} \cdot \left[\frac{1}{\varepsilon_0 \mu_0} \nabla \times \mathbf{B} - \frac{1}{\varepsilon_0} \mathbf{j} \right] + \frac{1}{\mu_0} \int d^3x \, \mathbf{B} \cdot (-\nabla \times \mathbf{E}) = \\ & - \int d^3x \, \mathbf{j} \cdot \mathbf{E} + \frac{1}{\mu_0} \int d^3x \, \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \frac{1}{\mu_0} \int d^3x \, \mathbf{B} \cdot (\nabla \times \mathbf{E}). \end{aligned}$$

Using the vector identity:

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}),$$

we can write:

$$\int d^3x \Big[\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E}) \Big] = - \int d^3x \nabla \cdot (\mathbf{E} \times \mathbf{B}) = \int_S (\mathbf{E} \times \mathbf{B}) \cdot d\vec{\sigma} = 0;$$

at infinity, the fields approach zero to keep a finite energy.
Finally, we obtain:

$$\begin{aligned} \frac{dE_{tot}}{dt} &= \frac{dE_p}{dt} + \frac{dE_f}{dt} = 0 \quad \text{The total energy is conserved} \\ \frac{dE_f}{dt} &= -\frac{dE_p}{dt} = - \int d^3x \, \mathbf{j} \cdot \mathbf{E} \quad \text{Work of the field } \ \mathbf{E} \end{aligned}$$