Plasma Physics I

Solution to the Series 5 (October 12, 2024)

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Exercise 1

a) To show that the magnetic flux is frozen in the plasma (described by ideal MHD), we will use the Ohm's law:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \tag{1}$$

and two Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

Let us consider the magnetic flux through a surface S that moves with the plasma. This surface is bound to the plasma and varies when the plasma moves. We have

$$\Phi_B(t) \equiv \int\int_{S(t)} \mathbf{B}(t) \cdot \mathbf{dS}$$

We consider a volume integral of $\nabla \cdot \mathbf{B}$ for $\mathbf{B}$-field at $t + \delta t$ bounded by the surfaces $S(t)$, $S(t + \delta t)$, and S_{side} (see Fig. 1):

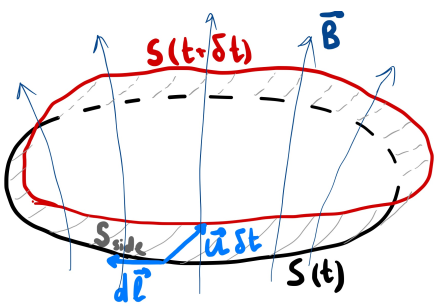

Figure 1: Sketch illustrating the integration volume and its boundary surfaces.

$$
\begin{aligned}
&\iiint \nabla \cdot \mathbf{B}(\mathbf{r}, t+\delta t) = 0 \\
&= \iint_{S(t)} \underbrace{\mathbf{B}(\mathbf{r}, t+\delta t)}_{\approx \mathbf{B}(\mathbf{r},t) + \frac{\partial \mathbf{B}}{\partial t}\delta t} \cdot \mathbf{dS} \;+\; \underbrace{\iint_{S(t+\delta t)} \mathbf{B}(\mathbf{r}, t+\delta t) \cdot \mathbf{dS}}_{= \Phi(t+\delta t)} \;+\; \iint_{Side} \mathbf{B}(\mathbf{r}, t+\delta t) \cdot \mathbf{dS} \\
&\approx \underbrace{\iint_{S(t)} \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS}}_{= -\Phi(t)} \;+\; \delta t \iint_{S(t)} \frac{\partial \mathbf{B}}{\partial t}(\mathbf{r}, t) \cdot \mathbf{dS} \;+\; \Phi(t+\delta t) \;+\; \iint_{Side} \mathbf{B}(\mathbf{r}, t+\delta t) \cdot \mathbf{dS} \\
&= -\Phi(t) \;+\; \delta t \iint_{S(t)} \frac{\partial \mathbf{B}}{\partial t}(\mathbf{r}, t) \cdot \mathbf{dS} \;+\; \Phi(t+\delta t) \;+\; \iint_{Side} \mathbf{B}(\mathbf{r}, t+\delta t) \cdot \mathbf{dS}
\end{aligned} \tag{4}
$$

as illustrated in Fig. 1, the *side-surface* integral is given by:

$$
\begin{aligned}
\iint_{Side} \mathbf{B}(\mathbf{r}, t+\delta t) \cdot \mathbf{dS} &= \oint_{\partial S(t) \equiv C} \mathbf{B}(\mathbf{r}, t+\delta t) \cdot (\delta t \mathbf{u} \times \mathbf{dl}) \\
&= -\oint_C \delta t \; (\mathbf{u} \times \mathbf{B}(\mathbf{r}, t+\delta t)) \cdot \mathbf{dl} \\
&\approx -\delta t \oint_C (\mathbf{u} \times \mathbf{B}(\mathbf{r}, t)) \cdot \mathbf{dl} \;-\; \delta t^2 \oint_C \left(\mathbf{u} \times \frac{\partial \mathbf{B}}{\partial t}(\mathbf{r}, t) \right) \cdot \mathbf{dl} \\
&\approx -\delta t \oint_C (\mathbf{u} \times \mathbf{B}(\mathbf{r}, t)) \cdot \mathbf{dl}
\end{aligned} \tag{5}
$$

Using equations (4) and (5) we get:

$$
\begin{aligned}
\delta\Phi &= \Phi(t+\delta t) - \Phi(t) \\
&= \delta t \left(-\iint_{S(t)} \underbrace{\frac{\partial \mathbf{B}}{\partial t}(\mathbf{r}, t)}_{= -\nabla \times \mathbf{E}} \cdot \mathbf{dS} \;+\; \oint_C (\mathbf{u} \times \mathbf{B}(\mathbf{r}, t)) \cdot \mathbf{dl} \right) \\
&= \delta t \left(\underbrace{\iint_{S(t)} (\nabla \times \mathbf{E}(\mathbf{r}, t)) \cdot \mathbf{dS}}_{= \oint_C \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{dl}} \;+\; \oint_C (\mathbf{u} \times \mathbf{B}(\mathbf{r}, t)) \cdot \mathbf{dl} \right) \\
&= \delta t \oint_C \underbrace{(\mathbf{E} + \mathbf{u} \times \mathbf{B})}_{= 0} \cdot \mathbf{dl} = 0
\end{aligned} \tag{6}
$$

were, in the last step, we used the Ohm's law of ideal MHD. Therefore, the magnetic flux is *frozen* in the plasma (the magnetic field lines will follow the plasma motion and vice-versa).

b) In the resistive MHD model the only equation that we used here that is modified is Ohm's law,

$$
\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}
$$

Therefore in this case we have

$$\frac{d\Phi_B}{dt} = -\int_C \eta \mathbf{J} \cdot \mathbf{dl} = -\int_C \frac{\eta}{\mu_0} (\nabla \times \mathbf{B}) \cdot \mathbf{dl} = \int\int_S \frac{\eta}{\mu_0} \nabla^2 \mathbf{B} \cdot \mathbf{dS} \neq 0$$

where we have used Ampere's law and Stokes's theorem.

c) We are looking for an equation to describe the evolution (in time) of the magnetic field $\mathbf{B}$. It's natural to start from the Faraday equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \tag{7}$$

We write $\mathbf{E}$ as a function of $\mathbf{u}$ and $\mathbf{J}$ (Ohm's law) and $\mathbf{J}$ as a function of $\mathbf{B}$ (Ampère's law):

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J} = -\mathbf{u} \times \mathbf{B} + \frac{\eta}{\mu_0} \nabla \times \mathbf{B}. \tag{8}$$

From this equation we have (we suppose η constant):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{\eta}{\mu_0} \nabla \times (\nabla \times \mathbf{B})$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{\eta}{\mu_0} \big(\nabla(\underbrace{\nabla \cdot \mathbf{B}}_{=0}) - \nabla^2 \mathbf{B}\big)$$

and finally:

$$\frac{\partial \mathbf{B}}{\partial t} - \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{convection}} = \underbrace{\frac{\eta}{\mu_0} \nabla^2 \mathbf{B}}_{\text{diffusion}} \tag{9}$$

The eq.(9) is the equation of the magnetic field transport in a plasma due to the resistivity raising from the collisions. The term η/μ_0 can be seen as a diffusion coefficient D.

The characteristic time for the diffusion of the magnetic field $\mathbf{B}$ in the plasma can be estimated from $D \sim L^2/\tau$:

$$\tau = \left(\frac{L^2 \mu_0}{\eta}\right) \tag{10}$$

We can estimate the resistivity using the Spitzer's formula: $\eta \simeq 8.72 \times 10^{-10}$ Ωm:

$$\tau \simeq \frac{3^2 \cdot 4\pi \cdot 10^{-7}}{8.72 \times 10^{-10}} \simeq 13000 \text{ s}$$

which is an extraordinary long time. Some Tokamak observations (sawtooth crashes) or astrophysical phenomena display a diffusion of the magnetic field at a higher rate than expected. This is usually believed to be due to collective phenomena that enhance the plasma resistivity (anomalous resistivity).

Exercise 2

a) Consider a transverse wave in a string with tension S and mass per unit length M (figure 2).

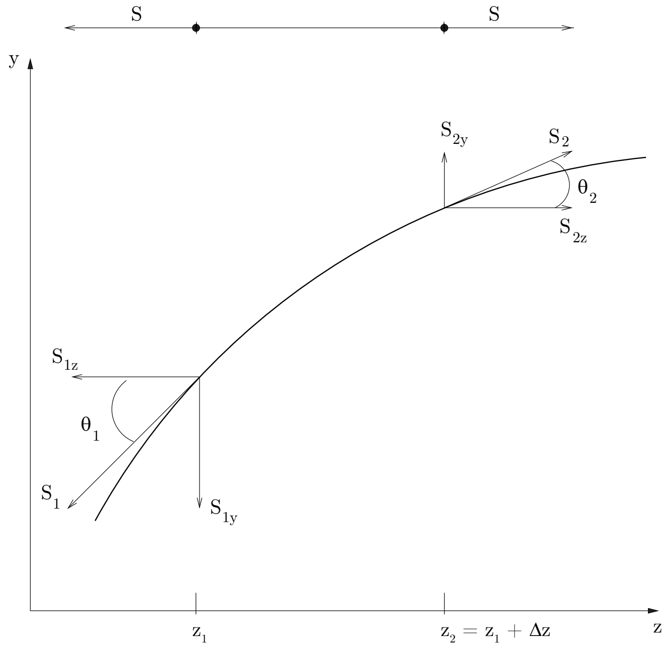

Figure 2: Transverse wave in a string.

For a purely transverse wave the net force along the z direction vanishes:

$$|S_{1z}| = |S_{2z}| = S \tag{11}$$

Considering the geometry of the problem:

$$\frac{S_{1y}}{S_{1z}} = \frac{S_{1y}}{S} = \tan\theta_1 = -\left(\frac{\partial y}{\partial z}\right)\Big|_z \tag{12}$$

$$\frac{S_{2y}}{S} = \tan\theta_2 = \left(\frac{\partial y}{\partial z}\right)\Big|_{z+\Delta z} \tag{13}$$

Using Newton's law in the y direction:

$$F_y^{tot} = m\frac{\partial^2 y}{\partial t^2} \Rightarrow S\left[\left(\frac{\partial y}{\partial z}\right)\Big|_{z+\Delta z} - \left(\frac{\partial y}{\partial z}\right)\Big|_z\right] = M\Delta z\,\frac{\partial^2 y}{\partial t^2} \tag{14}$$

we have:

$$S\frac{\left(\frac{\partial y}{\partial z}\right)\Big|_{z+\Delta z} - \left(\frac{\partial y}{\partial z}\right)\Big|_z}{\Delta z} = M\,\frac{\partial^2 y}{\partial t^2} \tag{15}$$

and, considering the limit $\Delta z \to 0$

$$S\frac{\partial^2 y}{\partial z^2} = M\frac{\partial^2 y}{\partial t^2} \Rightarrow \frac{\partial^2 y}{\partial z^2} = \frac{M}{S}\,\frac{\partial^2 y}{\partial t^2} \tag{16}$$

This is the equation of a wave with velocity v given by:

$$v = \sqrt{\frac{S}{M}} \tag{17}$$

b) We are considering small perturbations to a uniform equilibrium:

$$\mathbf{u}_0 = 0; \;\; \rho_0, \;\; p_0 \text{ uniforms}; \;\; \mathbf{B}_0 = B_0 \mathbf{e}_z \tag{18}$$

The linearised ideal MHD equations with respect to that equilibrium are:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}_1 = 0 \tag{19}$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla p_1 + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 \tag{20}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \tag{21}$$

$$\frac{\partial p_1}{\partial t} = c_s^2 \frac{\partial \rho_1}{\partial t} \tag{22}$$

We use the same geometry that has been considered during the course, with a magnetic field along z, $\mathbf{B} = B_0 \hat{e}_z$ and the velocity perturbation in the y direction:

$$\mathbf{B_0} = (0, 0, B_0)$$
$$\mathbf{u}_1 = (0, u_{1y}, 0)$$
$$\mathbf{k} = (0, 0, k_z)$$

Note: we don't use Fourier; nevertheless, having $k_x = k_y = 0$ in the real space the quantities are not changing in the x and y directions ($\partial/\partial x = \partial/\partial y = 0$).

Since $\partial/\partial y = 0$ and $\mathbf{u}_1 = (0, u_{1y}, 0)$, we have $\nabla \cdot \mathbf{u}_1 = 0$, and therefore $\rho_1 = 0$ and $\frac{\partial p_1}{\partial t} = c_s^2 \frac{\partial \rho_1}{\partial t} = 0$, and hence, $p_1 = 0$. Rewriting the cross products:

$$\mathbf{u}_1 \times \mathbf{B_0} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 0 & u_{1y} & 0 \\ 0 & 0 & B_0 \end{vmatrix} = u_{1y} B_0 \hat{e}_x$$

$$\nabla \times (\mathbf{u}_1 \times \mathbf{B_0}) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ u_{1y} B_0 & 0 & 0 \end{vmatrix} = B_0 \frac{\partial u_{1y}}{\partial z} \hat{e}_y$$

$$(\nabla \times \mathbf{B}_1) \times \mathbf{B_0} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -\frac{\partial B_{1y}}{\partial z} & \frac{\partial B_{1x}}{\partial z} & 0 \\ 0 & 0 & B_0 \end{vmatrix} = B_0 \frac{\partial B_{1x}}{\partial z} \hat{e}_x + \quad B_0 \frac{\partial B_{1y}}{\partial z} \hat{e}_y$$

The system of equations is then:

$$\rho_0 \frac{\partial u_{1y}}{\partial t} = \frac{B_0}{\mu_0} \frac{\partial B_{1y}}{\partial z} \tag{23}$$

$$\frac{\partial B_{1y}}{\partial t} = B_0 \frac{\partial u_{1y}}{\partial z} \tag{24}$$

If we consider the time derivative of the eq.(24):

$$\frac{\partial^2 B_{1y}}{\partial t^2} = B_0 \frac{\partial^2 u_{1y}}{\partial z \partial t}; \tag{25}$$

substituting the term $\partial u_{1y}/\partial t$ from the eq.(23) we find:

$$\frac{\partial^2 B_{1y}}{\partial t^2} = B_0 \frac{\partial}{\partial z}\left(\frac{B_0}{\mu_0 \rho_0}\frac{\partial B_{1y}}{\partial z}\right) = \frac{B_0^2}{\mu_0 \rho_0}\frac{\partial^2 B_{1y}}{\partial z^2}$$

$$\Rightarrow \quad \frac{\partial^2 B_{1y}}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho_0}\frac{\partial^2 B_{1y}}{\partial z^2}$$

that is the equation of a wave propagating at the Alfvén velocity:

$$c_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}} \tag{26}$$

formally equivalent to the equation for a transvere wave on a string with tension S and mass per unit length M:

$$v = \sqrt{\frac{S}{M}} \;\Leftrightarrow\; c_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}} \tag{27}$$

Therefore we can associate B_0^2/μ_0 (magnetic pressure) to the tension S, and ρ_0 (plasma mass density) to the mass per unit length M.

c) If we consider a plasma in ITER:

$$c_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}} \quad \text{with} \quad \rho_0 = m_p(A_D n_D + n_T A_T) = m_p n_e (A_D/2 + A_T/2)$$

$$c_A = \frac{6}{\sqrt{4\pi \cdot 10^{-7} \cdot 1.67 \times 10^{-27} \cdot 10^{20} \cdot (1 + 1.5)}} \simeq 8.27 \times 10^6 \text{ ms}^{-1}$$

d) Which particles are resonant with the Alfvén waves? We need to estimate the velocity of the charged particles: electrons, D ions, D beam ions, T ions and α particles. We obtain
$v_{\text{th},D} \simeq 7.9 \times 10^5$ m/s,
$v_{D,beam} = \sqrt{\frac{2E_D}{2m_p}} \simeq 9.8 \times 10^6$ m/s,
$v_{\text{th},T} \simeq 6.4 \times 10^5$ m/s,
$v_{\text{th},e} \simeq 4.8 \times 10^7$ m/s
$v_\alpha = \sqrt{\frac{2E_\alpha}{4m_p}} \simeq 1.3 \times 10^7$ m/s.
Both the beam ions and α particles have a velocity that is initially higher than the Alfven velocity, so as they slow down via collisions with other particles (see exercise 3 of week 3 serie) they will resonate with the Alfven waves. The electron thermal velocity is much greater than the Alfven velocity, so some electrons do have the resonant velocity. However, we will see later in the course that if the gradient of the velocity distribution is small at the resonant velocity, no resonant effect occurs.