Plasma Physics I

Solution to the Series 4 (October 5, 2024)

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Exercise 1

We suppose to have uniform parameters (density, temperature, ...) in the considered volume. The relation between the electric field E and the current density j is $E = \eta j$. Having $I = j \cdot A$, we can write $I = \frac{A}{\eta} E$ where $A = \pi a^2$ is the section of the cylinder. The potential difference between the extremities of the cylinder is $\Delta V_{plasma} = E \cdot L$, therefore:

$$\Delta V_{plasma} = \frac{\eta I}{\pi a^2} L$$

Using the numerical values, we find:

$$E = \frac{I\eta}{\pi a^2} = \frac{Ze^2 \sqrt{m_e} 4\pi \ln \Lambda}{(4\pi\epsilon_0)^2 3\sqrt{2\pi} T_e^{3/2}} \frac{I}{\pi a^2} \simeq \frac{5.1 \times 10^{-5}\ Z\ \ln \Lambda}{(T_e[\mathrm{eV}])^{3/2}} \frac{I}{\pi a^2} \simeq 3 \times 10^{-3}\ \mathrm{V/m}$$

where the Spitzer resistivity (eq. 4.28 in the course) has been used. Here, $\ln \Lambda = 24 - ln(n_e^{1/2}(\mathrm{cm}^{-3})/T_e(\mathrm{eV})$.
Finally we have:

$$\Delta V_{plasma} = E \cdot L = 6\ \mathrm{mV}$$

If we consider a stainless steel cylinder with resistivity $\eta_{\mathrm{inox}} \approx 7 \times 10^{-7}\ \Omega$ m, we find:

$$E_{inox} = \frac{I\eta_{\mathrm{inox}}}{\pi a^2} = 7{\times}10^{-7} \frac{10^6}{0.25\ \pi}\ \mathrm{V/m} \approx 0.89\ \mathrm{V/m} \quad \Rightarrow \quad \Delta V_{\mathrm{inox}} = 1.78\ \mathrm{V} \gg \Delta V_{plasma}$$

Exercise 2

In the lecture, we have studied the behavior of the whole electron distribution function under the action of a strong electric field. We used the rather strong assumption of a drifting Maxwellian for the electrons. Here, we consider individual electrons of the high energy tail. For these electrons, besides collisions with ions, we also need to consider collisons with thermal electrons, such that $\nu = \nu_p^{e/e'} + \nu_p^{e/i}$, where

$$\nu_p^{e/e} = n_e \frac{e^4}{2\pi\epsilon_0^2} \frac{\ln \Lambda}{m_e^2 v_e^3}$$

and

$$\nu_p^{e/i} = n_i \frac{Z^2 e^4}{4\pi\epsilon_0^2} \frac{\ln \Lambda}{m_e m_e v_e^3}$$

Using the quasineutrality condition, $n_e = Zn_i$, we get the total collision frequency as

$$\nu = (2+Z)\frac{n_e e^4}{4\pi\epsilon_0^2}\frac{\ln\Lambda}{m_e^2 v^3}$$

The momentum equation for electrons at the velocity v is:

$$m_e\frac{dv}{dt} = -eE - \nu(v)m_e v = -eE - F_c(v)$$

Which is the sign of the right-side of this equation? If the friction force due to the collisions is bigger then the acceleration force due to the electric field ($F_c(v) > e|E|$), there is a deceleration until the thermal velocity is reached.

On the other hand, if $e|E| > F_c(v)$ there is an acceleration. When the velocity v increases, the friction force decrease ($F_c(v) \propto v^{-2}$) and the acceleration is amplified (run-away regime).

Which is the expression for the minimum electric field necessary to be in the run-away regime?

$$e|E| = F_c(v) = (2+Z)\frac{e^4}{4\pi\epsilon_0^2}n_e\frac{\ln\Lambda}{m_e^2 v^3}m_e v$$

and the critical kinetic energy $E_{k,crit} = \frac{m_e v_{crit}^2}{2}$ is:

$$E_{k,crit} = (2+Z)\frac{e^3}{8\pi\epsilon_0^2}n_e\ln\Lambda\frac{1}{|E|} = T_e\frac{E_D}{|E|}$$

where we have used the critical electric E_D field defined as:

$$E_D \equiv (2+Z)\frac{e^3}{8\pi\epsilon_0^2}\ln\Lambda\frac{n_e}{T_e}$$

This electric field is called *Dreicer field*.

The numerical value of the critical kinetic energy is:

$$E_{k,crit} = (2+1)\frac{(1.6\times10^{-19})^3}{8\pi\cdot(8.85\times10^{-12})^2}17\frac{10^{20}}{3.2\times10^{-3}} \simeq 3.3\times10^{-12}\text{ J} \simeq 21\text{ MeV}$$

Be careful with the notation: E_D is the critical electric field and $E_{k,crit}$ is the critical kinetic energy.

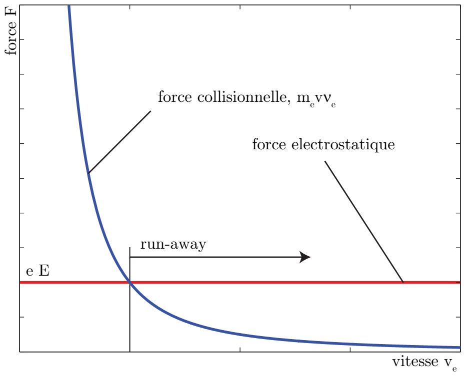

Figure 1: Force due to the electric field E and force due to the collisions for an electron with velocity v_e.

Exercise 3

a) Particle transport across the magnetic field lines could be explained by diffusion due to charge particle collisions. However, independently of the value of the diffusion coefficient, the measured density profile is not compatible with the classic diffusion. To show it, we can use the continuity equation:

$$\frac{\partial n}{\partial t} - \nabla \cdot (D_\perp \nabla n) = S(r).$$

Here, $S(r)$ is the source term due to the neutral beam injection or electromagnetic waves, that for simplicity we assume to be a radially symmetric, smooth function. In stationary conditions ($\partial n/\partial t = 0$), we have $-\nabla \cdot (D_\perp \nabla n) = S(r)$, therefore, in radial coordinates:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[rD_\perp(r)\frac{\partial n}{\partial r}\right] = S(r)$$

or, equivalently

$$-\frac{\partial}{\partial r}\left[rD_\perp(r)\frac{\partial n}{\partial r}\right] = rS(r)$$

Integrating this expression from $r = 0$ to r_0, we find

$$-\int_0^{r_0} \frac{\partial}{\partial r}\left[rD_\perp(r)\frac{\partial n}{\partial r}\right] dr = -r_0 D_\perp(r_0)\left.\frac{\partial n}{\partial r}\right|_{r=r_0} = \int_0^{r_0} rS(r)dr = A{\cdot}\bar{S}(r_0)/2\pi > 0$$

where $\bar{S}(r_0)$ can be seen as an average value of the source from $r = 0$ to $r = r_0$ and A is a surface.

Finally

$$D_\perp(r_0) = -\frac{A\bar{S}(r_0)}{r_0 \left.\frac{\partial n}{\partial r}\right|_{r=r_0}}$$

The density profile is flat for $0 \leq r \leq 0.5$ m, therefore $\frac{\partial n}{\partial r} = 0$ and $D_\perp \to \infty$ there. Moreover, at $r = 0.5$ m the gradient $\frac{\partial n}{\partial r}$ becomes finite and $D_\perp$ has a discontinuity that is not possible to justify using the classical theory.

b) An effective diffusion coefficient can be estimated at $r = 0.5$ m by considering the density gradient at this position. If we write

$$\Gamma_n = -D_{\text{eff}} \nabla n$$

we find $D_{\text{eff}} = \frac{\Gamma_n}{\partial n/\partial r} = \frac{8\times 10^{20}}{10^{21}} = 0.8 \text{ m}^2\text{s}^{-1}$.

The characteristic length we should use to estimate the classical diffusion across the magnetic field lines is the Larmor radius. We need to consider the collision frequency for the momentum transfer between un-like particles:

$$D_{\perp,\alpha} \simeq \rho_{L,\alpha}^2 \bar{\nu}_p^{\alpha/\beta} \tag{1}$$

In this case, the diffusion is automatically ambipolar because $D_{\perp,e} \simeq D_{\perp,i} = D_\perp$.

If we consider a hydrogen plasma, the collision frequency for electrons is:

$$\rho_{L,e} \text{ [m]} = \frac{\sqrt{m_e T_e}}{|e|B} \simeq 2.38 \times 10^{-6} \frac{\sqrt{T_e \text{ [eV]}}}{B \text{ [T]}} \simeq 1.2 \times 10^{-4} \text{ m}$$

$$\bar{\nu}_p^{e/i} \text{ [s}^{-1}\text{]} = \frac{1}{3}\sqrt{\frac{2}{\pi}} \frac{n_i Z^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_e^{1/2}} \frac{1}{T_e^{3/2}} = 4.41\times 10^{-11} \frac{n_i \text{ [m}^{-3}\text{]}}{T_e^{3/2} \text{ [eV]}} = 4.41\times 10^{3} \text{ s}^{-1}$$

therefore, the diffusion coefficient is:

$$D_\perp \simeq (1.2 \times 10^{-4})^2 \cdot 4.41 \times 10^3 \simeq 6.25 \times 10^{-5} \text{ m}^2\text{s}^{-1} \ll D_{\text{eff}} \tag{2}$$

c) The mechanism for particle transport in a tokamak is clearly not classical diffusion. Even neoclassical diffusion (which takes into account corrections due to toroidal effects on particle orbits and which has not been treated during the lecture) is not enough at all to explain such high level of transport, i.e. $D_{\text{eff}} \sim 1$. This anomalous transport is attributed to collective effects in the plasma dynamics, which produce turbulent flows of particles and heat.