Plasma I

Solution to the Series 2 (September 21, 2024)

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Exercise 1

a) The *continuity equation* for the density is:

$$\frac{\partial n}{\partial t} + \underline{\nabla} \cdot \mathbf{\Gamma} = S - \alpha n^2 \tag{1}$$

where S is the source term that ensures $\frac{\partial n}{\partial t} = 0$ and $-\alpha n^2$ is the loss due to the recombination process. Using the Fick's law:

$$\mathbf{\Gamma} = -D\underline{\nabla} n \tag{2}$$

We can express the *global* transport in the volume V as:

$$\underbrace{\int_V \frac{\partial n}{\partial t} d^3r}_{\frac{dN}{dt}} + \underbrace{\int_V \underline{\nabla} \cdot (-D\underline{\nabla} n) d^3r}_{\int_{\partial V} d^2\mathbf{s} \cdot (-D\underline{\nabla} n)} = \int_V S d^3r - \alpha \int_V n^2 d^3r \tag{3}$$

The integral $\int_{\partial V}$ (Gauss's theorem) is done over the boundary of the considered volume and $\mathbf{s}$ is the outward-pointing unit vector normal to the surface. Due to the slab geometry (figure 1), we only have one spatial independent variable (x):

$$\underline{\nabla} n \equiv \frac{dn}{dx}\hat{x}$$

therefore the integral along the directions y and z is a constant A (unknown) equal to the surface through which the flux is present:

$$\int_{\partial V} d^2\mathbf{s} \cdot (-D\underline{\nabla} n) = -AD\left(\left.\frac{dn}{dx}\right|_L - \left.\frac{dn}{dx}\right|_{-L}\right) \tag{4}$$

The equation (3) can be rewritten as:

$$\frac{dN}{dt} = AD\left.\frac{dn}{dx}\right|_{-L}^{L} + A\int_{-L}^{L} S\,dx - \alpha A \int_{-L}^{L} n^2 dx = 0 \tag{5}$$

with

$$\left.\frac{dn}{dx}\right|_{-L}^{L} = -\frac{n_0\pi}{2L}\sin\left(\frac{\pi x}{2L}\right)\Big|_{-L}^{L} = -\frac{n_0\pi}{L} \tag{6}$$

and, using the trigonometric identity $\cos^2\xi = 1/2\left[1 + cos(2\xi)\right]$

$$\int_{-L}^{L} n^2 dx = n_0^2 \int_{-L}^{L} \cos^2\left(\frac{\pi x}{2L}\right) dx = n_0^2 \int_{-L}^{L} \frac{1}{2}\left[1 + \cos\left(\frac{\pi x}{L}\right)\right] dx = n_0^2 L \tag{7}$$

$$\Rightarrow \int_{-L}^{L} S dx = \alpha n_0^2 L + \frac{D n_0 \pi}{L} \tag{8}$$

b) The value of the density $\tilde{n}_0$ necessary to have a loss rate due to the diffusion equal to that due to the recombination is:

$$\frac{D\tilde{n}_0\pi}{L} = \alpha\tilde{n}_0^2 L \rightarrow \tilde{n}_0 = \frac{\pi D}{\alpha L^2} = \frac{\pi \cdot 0.1 \text{ m}^2/\text{s}}{10^{-15} \text{ m}^3/\text{s} \cdot (2 \text{ m})^2} \approx 8\times 10^{13} \text{ m}^{-3} \tag{9}$$

When the density is bigger than $\tilde{n}_0$ the recombination process is the dominant one.

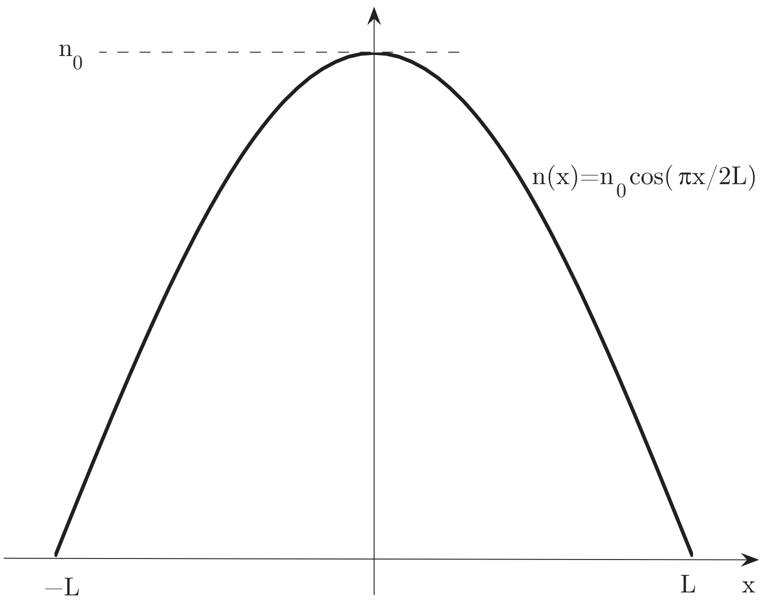

Figure 1: Density profile for a slab geometry

c) To estimate the confinement time τ_p we can use the following relation:

$$\frac{N_0}{\tau_p} = \left.\frac{dN}{dt}\right|_{\text{lost}} \tag{10}$$

with

$$N_0 = \int_V n d^3 r = A n_0 \int_{-L}^{L} \cos\left(\frac{\pi x}{2L}\right) dx = A n_0 \frac{2L}{\pi} \sin\left(\frac{\pi x}{2L}\right)\Bigg|_{-L}^{L} = A n_0 \frac{4L}{\pi} \tag{11}$$

and therefore

$$\tau_p = \frac{4Ln_0/\pi}{\alpha n_0^2 L + D n_0 \pi/L} = \frac{4/\pi}{\alpha n_0 + D\pi/L^2} = \frac{4/\pi}{10^{-15}\cdot 8\times 10^{13} + \frac{0.1\pi}{2^2}} \approx 8 \text{ s}, \qquad \text{for } n_0 = \tilde{n}_0 \tag{12}$$

Exercise 2

a) The process can be described as an ambipolar diffusion process, with diffusion coefficient

$$D_a \approx 2D_i = 2\frac{v_{thi}^2}{\nu_{i/n}} \tag{13}$$

where $\nu_{i/n} = n_n \sigma_{i/n} v_{thi}$ is the frequency of ion/neutral collisions. Therefore we have

$$D_a \approx 2\frac{v_{thi}}{n_n \sigma_{i/n}} \tag{14}$$

b) The continuity equation describing this process writes as

$$\frac{\partial n}{\partial t} - \underline{\nabla} \cdot (D\underline{\nabla} n) = S$$

Therefore, since the system has cylindrical symmetry, with only a radial dependence ($\partial/\partial\theta = \partial/\partial z = 0$), the diffusion equation reads, in stationary conditions, as

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[rD(r)\frac{\partial n}{\partial r}\right] = S(r)$$

where $S(r) = S_0$ for $r \leq r_s$ and $S(r) = 0$ for $r > r_s$. The diffusion coefficient is assumed to be constant, $D(r) = D_a$. We first solve for the density in the source region. Integrating this expression from $r = 0$ to r,

$$\int_0^r \frac{\partial}{\partial r}(r\frac{\partial n}{\partial r})dr = -\frac{S_0}{D_a}\frac{r^2}{2}$$

and imposing the regularity condition $r\frac{\partial n}{\partial r}\Big|_{r=0} = 0$, we find

$$n(r) = n(0) - \frac{S_0}{D_a}\frac{r^2}{4} \quad \text{for} \quad 0 < r \leq r_s$$

where $n(0)$ is unknown for the moment. Now we solve for the density in the outer region,

$$\frac{\partial}{\partial r}(r\frac{\partial n}{\partial r}) = 0$$

Therefore

$$\frac{\partial n}{\partial r} = \frac{C}{r}$$

where $C = r_s\frac{\partial n}{\partial r}\Big|_{r_s}$. Integrating this expression from $r = r_s$ to r, we get

$$n(r) = n(r_s) + r_s\frac{\partial n}{\partial r}\Big|_{r_s} \ln\left(\frac{r}{r_s}\right) \quad \text{for} \quad r_s < r \leq r_c$$

Imposing the boundary condition $n(r_c) = 0$ and the matching conditions at $r = r_s$,

$$\lim_{r \to r_s^+} n(r) = \lim_{r \to r_s^-} n(r)$$

$$\lim_{r \to r_s^+} \frac{\partial n}{\partial r} = \lim_{r \to r_s^-} \frac{\partial n}{\partial r}$$

we get the general solution for the density profile,

$$n(r) = \frac{S_0}{D_a} \frac{r_s^2}{2} \left[\frac{1}{2} - \ln \left(\frac{r_s}{r_c} \right) \right] - \frac{S_0}{D_a} \frac{r^2}{4} \qquad \text{for} \quad r \le r_s$$

$$n(r) = -\frac{S_0}{D_a} \frac{r_s^2}{2} \ln \left(\frac{r}{r_c} \right) \qquad \text{for} \quad r \ge r_s$$

c) At $r = 0$ we have

$$n(0) = \frac{S_0}{D_a} \frac{r_s^2}{2} \left[\frac{1}{2} - \ln \left(\frac{r_s}{r_c} \right) \right] = \frac{S_0 n_n \sigma_{i/n}}{2 v_{thi}} \frac{r_s^2}{2} \left[\frac{1}{2} - \ln \left(\frac{r_s}{r_c} \right) \right]$$

Therefore

$$\frac{n(0)}{n_n} = \frac{S_0 \sigma_{i/n}}{2 v_{thi}} \frac{r_s^2}{2} \left[\frac{1}{2} - \ln \left(\frac{r_s}{r_c} \right) \right]$$

and the relative ionization degree at the center of the column is

$$\alpha = \frac{n(0)}{n_n + n(0)} = \frac{n(0)/n_n}{1 + n(0)/n_n}$$

The numerical application gives $\alpha \lesssim 10^{-4}$.