

Plasma Physics I

Solution to the Series 14 (December 21, 2024)

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Exercise 1

Consider an unmagnetized plasma described by the Vlasov-Poisson model:

$$\frac{\partial f_{\alpha 1}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\alpha 1}}{\partial \mathbf{x}} + \frac{q_a}{m_a} \mathbf{E}_1 \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0 \quad (1)$$

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\varepsilon_0} \sum_{\alpha} q_{\alpha} \int f_{\alpha 1} d^3 v. \quad (2)$$

Under the electrostatic approximation, the electric field is given by $\mathbf{E} = -\nabla \phi$.

Using the Fourier transform applied to the spatial variables and rewriting the derivatives as $\frac{\partial}{\partial \mathbf{x}} = i\mathbf{k}$, we have:

$$\frac{\partial f_{\alpha 1}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} f_{\alpha 1} - i \frac{q_a}{m_a} \phi \mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0 \quad (3)$$

$$k^2 \phi = \frac{1}{\varepsilon_0} \sum_{\alpha} q_{\alpha} \int f_{\alpha 1} d^3 v. \quad (4)$$

The Laplace transform can be used for the time variable:

$$p \tilde{f}_{\alpha 1} - g_{\alpha} + i\mathbf{k} \cdot \mathbf{v} \tilde{f}_{\alpha 1} - i \frac{q_a}{m_a} \tilde{\phi} \mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0 \quad (5)$$

$$k^2 \tilde{\phi} = \frac{1}{\varepsilon_0} \sum_{\alpha} q_{\alpha} \int \tilde{f}_{\alpha 1} d^3 v, \quad (6)$$

where $g_{\alpha} = \mathcal{F}[f_{\alpha 1}(t=0)] = \mathcal{F}[\delta(\mathbf{x})\delta(\mathbf{v})] = \delta(\mathbf{v})/(2\pi)^3$ for the test particle q_T , otherwise $g_{\alpha} = 0$. Here $\mathcal{F}[\cdot]$ denotes the Fourier transform.

We find $\tilde{f}_{\alpha 1}$ from the eq.(5):

$$\tilde{f}_{\alpha 1} = i \frac{q_a}{m_a} \tilde{\phi} \frac{\mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}}{p + i\mathbf{k} \cdot \mathbf{v}} + \frac{g_{\alpha}}{p + i\mathbf{k} \cdot \mathbf{v}}. \quad (7)$$

We can now substitute the expression for $\tilde{f}_{\alpha 1}$ on the eq.(6), obtaining:

$$k^2 \tilde{\phi} = \sum_{\alpha} \frac{q_{\alpha}}{\varepsilon_0} \int i \frac{q_a}{m_a} \tilde{\phi} \frac{\mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}}{p + i\mathbf{k} \cdot \mathbf{v}} d\mathbf{v} + \sum_{\alpha} \frac{q_{\alpha}}{\varepsilon_0} \int d\mathbf{v} \frac{g_{\alpha}}{p + i\mathbf{k} \cdot \mathbf{v}}$$

and finally:

$$\underbrace{\tilde{\phi} \left(1 - \sum_{\alpha} \frac{iq_{\alpha}^2}{\varepsilon_0 m_{\alpha} k^2} \int d\mathbf{v} \frac{\mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}}{p + i\mathbf{k} \cdot \mathbf{v}} \right)}_{D(\mathbf{k}, ip)} = \frac{1}{(2\pi)^3} \frac{q_T}{\varepsilon_0 k^2} \underbrace{\int d\mathbf{v} \frac{\delta(\mathbf{v})}{p + i\mathbf{k} \cdot \mathbf{v}}}_{\frac{1}{p}}$$

From the following equation:

$$\tilde{\phi}(\mathbf{k}, p) = \frac{1}{(2\pi)^3} \frac{q_T}{\varepsilon_0 k^2} \frac{1}{D(\mathbf{k}, ip)}$$

using the inverse Laplace transform we can evaluate the time evolution of ϕ :

$$\phi(\mathbf{k}, t) = \frac{1}{2\pi i} \int_{p_0 - i\infty}^{p_0 + i\infty} \tilde{\phi}(k, p) e^{pt} dp = \sum \text{Res} \left(\tilde{\phi}(k, p) e^{pt}, p = p_k \right).$$

We are interested on the pole $p = 0$ (which corresponds to a stationary condition, $t \rightarrow \infty$), therefore:

$$\phi(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \frac{q_T}{\varepsilon_0 k^2} \frac{1}{D(\mathbf{k}, p=0)}.$$

To evaluate the term $D(\mathbf{k}, p)$ for $p = 0$,

$$\begin{aligned} D(\mathbf{k}, p=0) &= 1 - \sum_{\alpha} \frac{iq_{\alpha}^2}{\varepsilon_0 m_{\alpha} k^2} \int d\mathbf{v} \frac{\mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}}{p + i\mathbf{k} \cdot \mathbf{v}} \\ &= 1 - \sum_{\alpha} \frac{\omega_{p,\alpha}^2}{n_{\alpha} k^2} \int_{\mathcal{L}} du \frac{dF_{0,\alpha}}{du} \frac{1}{u - \frac{\omega}{k}} \Big|_{\frac{\omega}{k}=0} \\ &= 1 - \sum_{\alpha} \frac{\omega_{p,\alpha}^2}{n_{\alpha} k^2} \left[\text{P.V.} \int \frac{dF_{0\alpha}}{du} \frac{1}{u} du + i\pi \frac{dF_{0\alpha}}{du} \Big|_{u=0} \right] \\ &= 1 - \sum_{\alpha} \frac{\omega_{p,\alpha}^2}{n_{\alpha} k^2} \text{P.V.} \int du \left(-\frac{u}{v_{th\alpha}^2} \frac{F_{0\alpha}}{u} \right) \\ &= 1 + \frac{1}{(k^2 \lambda_{De}^2)} + \frac{1}{(k^2 \lambda_{Di}^2)} \\ &= 1 + \frac{1}{(k^2 \lambda_D^2)} \end{aligned} \tag{8}$$

where $\frac{1}{\lambda_D^2} = \sum_{\alpha} \frac{\omega_{p,\alpha}^2}{v_{th\alpha}^2}$ and we used $\int_{-\infty}^{\infty} F_{\alpha 0} du = n_{\alpha}$.

Finally:

$$\phi(\mathbf{k}) = \frac{1}{(2\pi)^3} \frac{q_T}{\varepsilon_0 k^2} \frac{1}{\left(1 + \frac{1}{(k\lambda_D)^2} \right)} = \frac{1}{(2\pi)^3} \frac{q_T}{\varepsilon_0} \frac{1}{k^2 + k_D^2} \tag{9}$$

where we define $k_D = \lambda_D^{-1}$.

We still need to compute the inverse Fourier transform of $\phi(k)$:

$$\phi(\mathbf{x}) = \int d\mathbf{k} \frac{1}{(2\pi)^3} \frac{q_T}{\varepsilon_0} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{k^2 + k_D^2} = \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^3} \int d\mathbf{k} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{k^2 + k_D^2}. \tag{10}$$

The integral can be solved in the spherical coordinate frame (for $\mathbf{k}$):

$$\begin{cases} k_x = k \sin \theta \cos \varphi \\ k_y = k \sin \theta \sin \varphi \\ k_z = k \cos \theta \end{cases}$$

with $d\mathbf{k} = k^2 dk \sin \theta d\theta d\varphi$.

We choose the direction of $\mathbf{x}$ such that $\mathbf{k} \cdot \mathbf{x} = kr \cos \theta$.

$$\begin{aligned} \phi(\mathbf{x}) &= \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi \frac{e^{ikr \cos \theta}}{k^2 + k_D^2} \\ &= \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^3} \int_0^\infty dk \frac{k^2}{k^2 + k_D^2} \int_0^\pi d\theta \sin \theta e^{ikr \cos \theta} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \end{aligned} \quad (11)$$

but

$$\begin{aligned} \int_0^\pi d\theta \sin \theta e^{kr \cos \theta} &= - \int_1^{-1} d(\cos \theta) e^{ikr \cos \theta} = \\ &= \frac{e^{ikr} - e^{-ikr}}{ikr} = \frac{2i \sin(kr)}{ikr} = \frac{2}{kr} \sin(kr) \end{aligned}$$

therefore

$$\phi(\mathbf{x}) = \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^2} \frac{2}{r} \int_0^\infty dk \frac{k}{k^2 + k_D^2} \sin(kr). \quad (12)$$

Using the polar notation for the complex quantities, we have $\sin(kr) = \text{Im}(e^{ikr})$:

$$\phi(\mathbf{x}) = \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^2} \frac{2}{r} \int_0^\infty dk \frac{\text{Im}\{ke^{ikr}\}}{k^2 + k_D^2}. \quad (13)$$

Due to the symmetry of the integrated function (even function):

$$\int_0^\infty dk \frac{\text{Im}\{ke^{ikr}\}}{k^2 + k_D^2} = \frac{1}{2} \int_{-\infty}^\infty dk \frac{\text{Im}\{ke^{ikr}\}}{k^2 + k_D^2}$$

we can write:

$$\phi(\mathbf{x}) = \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^2} \frac{1}{r} \int_{-\infty}^\infty dk \frac{\text{Im}\{ke^{ikr}\}}{k^2 + k_D^2} = \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^2} \frac{1}{r} \text{Im} \left\{ \int_{-\infty}^\infty dk \frac{ke^{ikr}}{k^2 + k_D^2} \right\}. \quad (14)$$

The function $\frac{ke^{ikr}}{k^2 + k_D^2}$ has two poles $k = \pm ik_D$.

To evaluate the integral, we use the residues theorem based on a contour $\mathcal{C}$ that goes along the real axis and then counterclockwise along the semicircle Γ ¹(see figure 1).

$$\int_{\mathcal{C}} \frac{ke^{ikr}}{k^2 + k_D^2} = \int_{-\infty}^\infty dk \frac{ke^{ikr}}{k^2 + k_D^2} + \underbrace{\int_{\Gamma} \frac{ke^{ikr}}{k^2 + k_D^2}}_{=0} \quad (15)$$

¹if we evaluate the integral clockwise along a semicircle (pole $-ik_D$ bounded by the contour), the integral diverges.

The integral along Γ vanishes because $\text{Im}(k) \rightarrow +\infty$. The integral along the contour $\mathcal{C}$ is given by the sum of the residues of the poles bounded by the integration path:

$$\begin{aligned} \int_{\mathcal{C}} \frac{ke^{ikr}}{k^2 + k_D^2} &= 2\pi i \text{Res} \left(\frac{ke^{ikr}}{k^2 + k_D^2}, k = +ik_D \right) \\ &= \lim_{k \rightarrow ik_D} 2\pi i \frac{ke^{ikr}(k - ik_D)}{(k + ik_D)(k - ik_D)} = 2\pi i \frac{ik_D e^{-k_D r}}{2ik_D} = i\pi e^{-r/\lambda_D} \end{aligned} \quad (16)$$

Finally we have:

$$\phi(\mathbf{x}) = \frac{q_T}{4\pi\epsilon_0} \frac{e^{-r/\lambda_D}}{r} \quad (17)$$

that is the same result we have found at the beginning of the course – see lecture 1, Eq.(1.7).

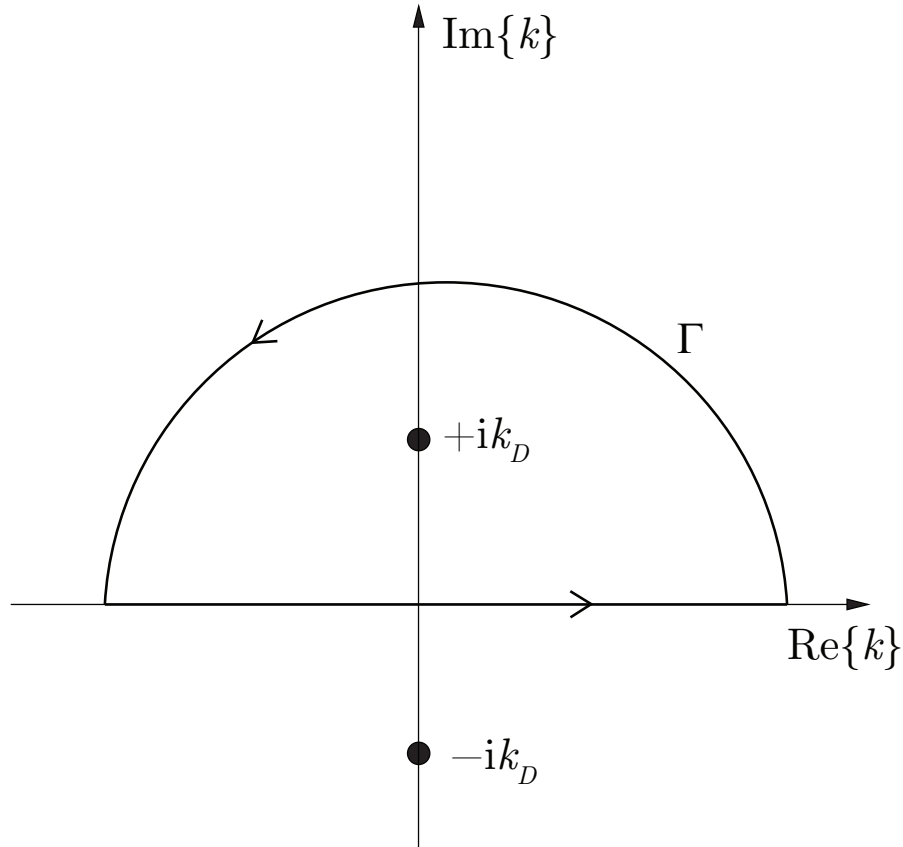


Figure 1: Integration path.