Plasma Physics I

Solution to the Series 13 (December 14, 2024)

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Exercise 1

The dispersion relations of waves in a magnetized plasma have been studied in the lectures. Using those results for a circular polarized wave [1] with $\mathbf{k} = (0, 0, k_z)$ and magnetic field $\mathbf{B}_0 = B_0\hat{e}_z$, we find the following dispersion relation:

$$N^2 = \epsilon_R = \epsilon_1 + \epsilon_2. \tag{1}$$

The expressions for ϵ_1 and ϵ_2 can be found in eq.(9.22) and eq.(9.23) of lecture 13. In the case of parallel propagation ($k_y = 0$), the argument of the Bessel function vanishes, therefore:

$$J_n(0) = \delta_{n,0}. \tag{2}$$

Only the terms $n = 0$ are present. The derivatives of Bessel functions $J_n'(a)$ and the terms $\frac{n}{a}J_n(a)$ in the tensor $\underline{\mathbf{T}}_\alpha$ can be rewritten as:

$$\begin{aligned} \frac{n}{a}J_n(a) &= \frac{1}{2}\left[J_{n-1}(a) + J_{n+1}(a)\right], \\ J_n'(a) &= \frac{1}{2}\left[J_{n-1}(a) - J_{n+1}(a)\right] \end{aligned}$$

Therefore, the tensor $\underline{\mathbf{T}}_\alpha$ is:

$$\underline{\mathbf{T}}_\alpha = \begin{pmatrix} \frac{v_\perp^2}{4}\left[\delta_{n,1} + \delta_{n,-1}\right] & i\frac{v_\perp^2}{4}\left[\delta_{n,1} - \delta_{n,-1}\right] & 0 \\ -i\frac{v_\perp^2}{4}\left[\delta_{n,1} - \delta_{n,-1}\right] & \frac{v_\perp^2}{4}\left[\delta_{n,1} + \delta_{n,-1}\right] & 0 \\ 0 & 0 & v_\parallel^2\delta_{n,0} \end{pmatrix} \tag{3}$$

Using the same notation as in lecture 6, we have:

$$\underline{\boldsymbol{\epsilon}} = \begin{pmatrix} \epsilon_1 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{4}$$

[1] We consider only right-hand circular polarization.

where:

$$\epsilon_1 = 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2}\left\{1 + \int d^3v \frac{v_\perp^2}{4\, n_{\alpha 0}}\left[\frac{k_z \frac{\partial f_{\alpha 0}}{\partial v_\parallel} + \frac{\Omega_\alpha}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp}}{k_z v_\parallel - \omega + \Omega_\alpha} + \frac{k_z \frac{\partial f_{\alpha 0}}{\partial v_\parallel} - \frac{\Omega_\alpha}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp}}{k_z v_\parallel - \omega - \Omega_\alpha}\right]\right\} \quad (5)$$

$$\epsilon_2 = \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2}\int d^3v \frac{v_\perp^2}{4\, n_{\alpha 0}}\left[\frac{k_z \frac{\partial f_{\alpha 0}}{\partial v_\parallel} + \frac{\Omega_\alpha}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp}}{k_z v_\parallel - \omega + \Omega_\alpha} - \frac{k_z \frac{\partial f_{\alpha 0}}{\partial v_\parallel} - \frac{\Omega_\alpha}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp}}{k_z v_\parallel - \omega - \Omega_\alpha}\right] \quad (6)$$

$$\epsilon_3 = 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2}\left\{1 + \int d^3v \frac{v_\parallel^2}{n_{\alpha 0}}\frac{k_z \frac{\partial f_{\alpha 0}}{\partial v_\parallel}}{k_z v_\parallel - \omega}\right\}. \quad (7)$$

The term ϵ_3 is related to longitudinal waves. The other two terms describe transverse waves and they can be rewritten as:

$$\epsilon_1 = 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2}\left[1 + I_1^+ + I_1^- + I_2^+ - I_2^-\right] \quad (8)$$

$$\epsilon_2 = \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2}\left[I_1^+ - I_1^- + I_2^+ + I_2^-\right] \quad (9)$$

where the following definition has been used:

$$I_1^\pm = \int d^3v \frac{v_\perp^2}{4\, n_{\alpha,0}}\left[\frac{k_z \frac{\partial f_{\alpha,0}}{\partial v_\parallel}}{k_z v_\parallel - \omega \pm \Omega_\alpha}\right]$$

$$I_2^\pm = \int d^3v \frac{v_\perp}{4\, n_{\alpha,0}}\left[\frac{\Omega_\alpha \frac{\partial f_{\alpha,0}}{\partial v_\perp}}{k_z v_\parallel - \omega \pm \Omega_\alpha}\right]$$

Our plasma is characterized by a Dirac delta distribution function:

$$f_{\alpha,0}(\mathbf{v}) = n_{\alpha,0}\,\delta(\mathbf{v}) \quad (10)$$

Solution of the integral $I_1^\pm$

$$
\begin{aligned}
I_1^\pm &= \int d^3v \frac{v_\perp^2}{4\,n_{\alpha,0}} \left[\frac{k_z \frac{\partial f_{\alpha,0}}{\partial v_\parallel}}{k_z v_\parallel - \omega \pm \Omega_\alpha} \right] \\
&= \frac{k_z}{4\,n_{\alpha,0}} \int dv_\perp 2\pi v_\perp^3 \int dv_\parallel \frac{\frac{\partial f_{\alpha,0}}{\partial v_\parallel}}{k_z v_\parallel - \omega \pm \Omega_\alpha} \\
&= \frac{k_z}{4\,n_{\alpha,0}} \int dv_\perp 2\pi v_\perp^3 \left[\underbrace{\frac{f_{\alpha,0}}{k_z v_\parallel - \omega \pm \Omega_\alpha}\Bigg|_{-\infty}^{+\infty}}_{=0} + \int dv_\parallel \frac{k_z f_{\alpha,0}}{(k_z v_\parallel - \omega \pm \Omega_\alpha)^2} \right] \\
&= \frac{k_z^2}{4\,n_{\alpha,0}} \int d^3v \frac{v_\perp^2 f_{\alpha,0}}{(k_z v_\parallel - \omega \pm \Omega_\alpha)^2} \\
&= \frac{k_z^2}{4} \int d^3v \frac{v_\perp^2 \delta(\mathbf{v})}{(k_z v_\parallel - \omega \pm \Omega_\alpha)^2} \\
&= 0
\end{aligned}
\tag{11}
$$

therefore the integral $I_1\pm$ does not give any contribution to the dispersion relation.

Solution of the integral $I_2^\pm$

$$
\begin{aligned}
I_2^\pm &= \frac{\Omega_\alpha}{4\,n_{\alpha,0}} \int d^3v\, v_\perp \left[\frac{\frac{\partial f_{\alpha,0}}{\partial v_\perp}}{k_z v_\parallel - \omega \pm \Omega_\alpha} \right] \\
&= \frac{\Omega_\alpha}{4\,n_{\alpha,0}} \left[\int \frac{dv_\parallel}{k_z v_\parallel - \omega \pm \Omega_\alpha} \int dv_\perp 2\pi v_\perp^2 \frac{\partial f_{\alpha,0}}{\partial v_\perp} \right] \\
&= \frac{\Omega_\alpha}{4\,n_{\alpha,0}} \left[\int \frac{dv_\parallel}{k_z v_\parallel - \omega \pm \Omega_\alpha} \left(\underbrace{v_\perp^2 f_{\alpha,0}\big|_0^{+\infty}}_{=0} - 2 \int dv_\perp 2\pi v_\perp f_{\alpha,0} \right) \right] \\
&= -\frac{\Omega_\alpha}{2\,n_{\alpha,0}} \int d^3v \frac{f_{\alpha,0}}{k_z v_\parallel - \omega \pm \Omega_\alpha} \\
&= -\frac{\Omega_\alpha}{2} \int d^3v \frac{\delta(\mathbf{v})}{k_z v_\parallel - \omega \pm \Omega_\alpha} = -\frac{\Omega_\alpha}{2(-\omega \pm \Omega_\alpha)} \int d^3v\, \delta(v_\parallel = 0, v_\perp = 0) \\
&= \frac{\Omega_\alpha/2}{\omega \mp \Omega_\alpha}
\end{aligned}
\tag{12}
$$

Dispersion relation

From the previous results, we find:

$$
\epsilon_1 = 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2} \left[1 + I_2^+ - I_2^- \right] \qquad \text{and} \qquad \epsilon_2 = \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2} \left[I_2^+ + I_2^- \right] \tag{13}
$$

and finally:

$$\begin{aligned}
N^2 = \epsilon_1 + \epsilon_2 &= 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2}\left[1 - 2I_2^-\right] \\
&= 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2}\left[1 - \frac{\Omega_\alpha}{\omega + \Omega_\alpha}\right] \\
&= 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2}\frac{\omega}{\omega + \Omega_\alpha} \\
&= 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega(\omega + \Omega_\alpha)}, \qquad (14)
\end{aligned}$$

that is the same relation that we have found from a fluid model with $T = 0$ (the Dirac delta distribution function can describe a particle population with no thermal velocity, therefore with $T = 0$).

Exercise 2

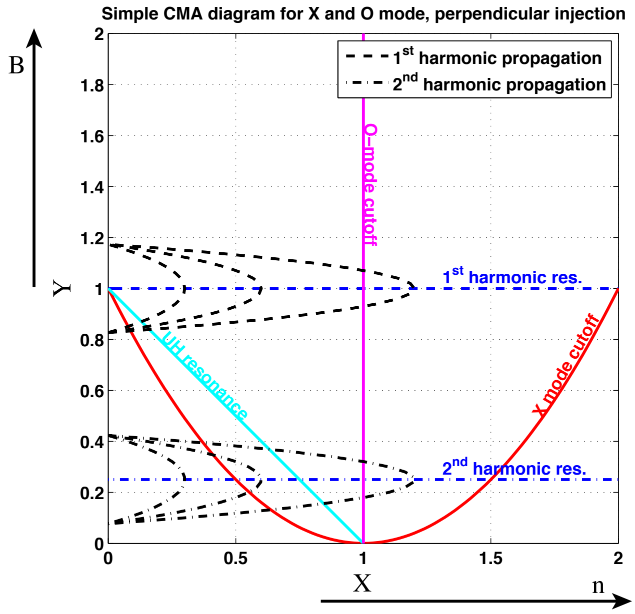

Figure 1: Clemmow-Mullaly-Allis diagram for X and O mode. Wave trajectories are shown for 1^st and 2^{nd} harmonic injection and for different core plasma densities. Note that for low field side X1 injection the wave first encounters a cutoff. X2 may encounter a cutoff or resonance, depending on the density. O mode has a higher density limit but will eventually be cut off at the plasma frequency.

a.) The X-mode cutoffs and resonances in terms of X, and Y are

- Cyclotron resonances $Y = 1/n^2$
- UH resonance: $1 = X + Y \rightarrow Y = 1 - X$
- Cutoff: $(\frac{\omega^2-\omega_p^2}{\omega^2})^2 - \frac{\Omega_e^2}{\omega^2} = 0 \rightarrow (1-X)^2 = Y$

b.) Consider the case $n = 1$ (first harmonic heating). Initially, the wave is outside the plasma so the density is zero $\rightarrow X = 0$. The field at the edge is lower than at the center so $\Omega_e < \Omega_{e0} \rightarrow Y < 1$. As the wave propagates from low field side to high field side, the magnetic field increases as $B \sim 1/r$. At the same time, the density increases (X increases). At the plasma center the density is highest and $B = B_0 \rightarrow \Omega_e = \Omega_{e0} \rightarrow Y = 1$). At the high field side the density is again zero ($X = 0$) and the field is higher than at the center, so $Y > 1$. For nth harmionic heating the picture is exactly the same but the values of Y are centered around $1/n^2$.

For first harmonic X-mode heating a wave launched from the LFS ($B < B_0$ so it starts *below* the resonance in the CMA diagram) first encounters the cutoff. It will therefore be reflected. However for 2nd harmonic heating and above it is possible for the wave to encounter the resonance first, providing the density is not too high. If it were possible to launch from the High Field Side (HFS), X1 heating would be possible as well.

c.) O-mode cutoffs and resonances in terms of X and Y:

- Cyclotron resonances $Y = 1/n^2$
- Cutoff: $X = 1$

O mode has fewer restrictions in terms of cutoff, the only cutoff being the plasma frequency which depends only on the density and is in any case at higher density than the X mode cutoff.

d.) Based on the magnetic fields the resonances are ITER:170 GHz, TCV:41 GHz. This rules out X2 heating on ITER because such frequencies are above the reach of present gyrotron technology. Indeed, ITER will use > 20 MW of EC heating in the first harmonic O mode (O1). 170 GHz gyrotron sources capable of continuously delivering 1 MW are being studied and developed at SPC-EPFL.

TCV on the other hand can in principle use X2 heating. In practice it uses both X2 (3 MW launched from the low field side) and X3 (1.5 MW launched from the top). The advantage of X3 is that higher density plumes can be heated than with X2.