Plasma Physics I

Solution to the Series 12 (December 7, 2024)

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Exercise 1

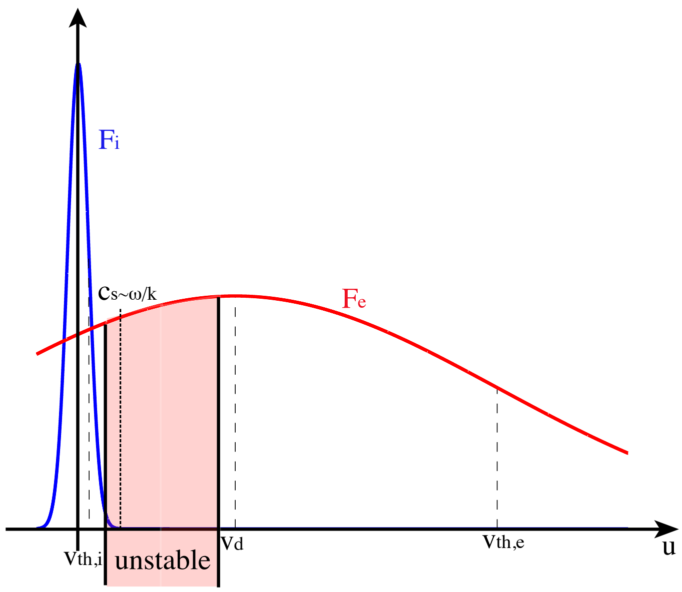

Figure 1: Plot of the distribution functions of ions and electrons.

a) We expect to have unstable waves where we have small ion damping and positive slope in the electron distribution function (Figure 1). Among the possible electrostatic waves that can exist in such plasma, only ion-acoustic waves have a phase velocity that falls into this region ($\omega \approx kc_s$). Any wave with a phase velocity smaller than c_s will be strongly damped by the ion distribution function.

b) From Landau theory, the damping rate γ is given by:

$$\gamma = -\frac{\epsilon_i(\omega_r)}{\partial\epsilon_r/\partial\omega_r}$$

where we can identify $\epsilon(\omega, k) = \epsilon_r(\omega, k) + i\epsilon_i(\omega, k)$ and the terms ω_r and γ, that are respectively the real part and the imaginary part of frequency:

$$\omega = \omega_r + i\gamma. \tag{1}$$

Real part of $\epsilon(\omega_r, k)$

The real part of ϵ is given by:

$$\epsilon_r(\omega_r, k) = 1 + \sum_\alpha \frac{e^2}{m_\alpha \epsilon_0 k} \text{P.V.} \int_{-\infty}^{\infty} \frac{dF_{0,\alpha}}{du} \frac{1}{\omega_r - ku} du \tag{2}$$

We can follow the solution of the ex. 2, series 11 and ex. 2, series 9.

For the ions, there are no differences in the solution:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}F_{0,i}}{\mathrm{d}u} \frac{1}{\omega_r - ku} du \approx -\frac{kn_i}{\omega_r^2} \tag{3}$$

For the electrons, we have ($\omega_r/k \ll v_{th,e}$):

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}F_{0,e}}{\mathrm{d}u} \frac{1}{\omega_r - ku} du \approx -\frac{1}{k} \int_{-\infty}^{\infty} \frac{\mathrm{d}F_{0,e}}{\mathrm{d}u} \frac{du}{u} \tag{4}$$

where

$$\frac{\mathrm{d}F_{0,e}}{\mathrm{d}u} = -\frac{u - v_d}{v_{\mathrm{th},e}^2} F_{0,e} \tag{5}$$

We obtain:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}F_{0,e}}{\mathrm{d}u} \frac{1}{\omega - ku} du \approx \frac{1}{kv_{\mathrm{th},e}^2} \left\{ \int_{-\infty}^{\infty} u\, F_{0,e} \frac{du}{u} - \int_{-\infty}^{\infty} v_d\, F_{0,e} \frac{du}{u} \right\} \approx \int_{-\infty}^{\infty} u\, F_{0,e} \frac{du}{u} = \frac{n_e}{kv_{\mathrm{th},e}^2} \tag{6}$$

where, due to the symmetry of $F_{0,e}$ (assuming $v_d \ll v_{th,e}$), $\int_{-\infty}^{\infty} v_d\, F_{0,e} \frac{du}{u} \approx 0$.

The real part of $\epsilon(\omega_r, k)$ is therefore given by:

$$\epsilon_r(\omega_r, k) \simeq 1 - \frac{\omega_{pi}^2}{\omega_r^2} + \frac{\omega_{pe}^2}{k^2 v_{th,e}^2},$$

and the derivative of ϵ_r with respect to ω_r is:

$$\frac{\partial \epsilon_r(\omega_r, k)}{\partial \omega_r} = 2\frac{\omega_{pi}^2}{\omega_r^3} \tag{7}$$

Imaginary part of $\epsilon(\omega_r, k)$

The imaginary part of ϵ is given by:

$$\epsilon_i(\omega_r, k) = -\pi \sum_\alpha \frac{e^2}{m_\alpha \varepsilon_0 k^2} \left. \frac{dF_{0,\alpha}}{du} \right|_{u=\omega/k} \tag{8}$$

Following Ex. 1 of series 10, for ions we don't have any difference:

$$\epsilon_i^{\text{ions}}(\omega_r, k) = \sqrt{\frac{\pi}{2}} \frac{c_s}{k^2} \frac{\omega_{pi}^2}{v_{th,i}^3} \exp\left(-\frac{c_s^2}{2\, v_{th,i}^2}\right) \tag{9}$$

For electrons:

$$\frac{\mathrm{d}F_{0,e}}{\mathrm{d}u} = -\frac{u - v_d}{v_{\mathrm{th},e}^2} F_{0,e} \tag{10}$$

where

$$F_{0,e} = \frac{n}{\sqrt{2\pi} v_{th,e}} \exp\left[-\frac{(u - v_d)^2}{2 v_{th,e}^2}\right]$$

By substituting the dispersion relation for the ions, $u = \omega_r/k = c_s$, and using the approximation $c_s \ll v_d$, we obtain:

$$\epsilon_i^{\text{electrons}}(\omega_r, k) = -\sqrt{\frac{\pi}{2}} \frac{v_d}{k^2} \frac{\omega_{pe}^2}{v_{th,e}^3} \exp\left(-\frac{v_d^2}{2\, v_{th,e}^2}\right). \tag{11}$$

Finally, we can derive the total damping rate:

$$\gamma = \gamma_e + \gamma_i = \sqrt{\frac{\pi}{8}} \left\{ k v_d \left(\frac{m_e}{m_i}\right)^{1/2} \exp\left[-\frac{v_d^2}{2 v_{th,e}^2}\right] - k c_s \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left[-\frac{T_e}{2 T_i}\right] \right\}$$

where we have introduced explicitly the expressions for the thermal velocities, the sound speed and the plasma frequencies.

The electron contribution is destabilizing ($\gamma_e > 0$), while the ion contribution is stabilizing (damping term $\gamma_i < 0$).

c) When $T_e \gg T_i$, the damping term from the ions contribution vanishes ($\gamma_i \to 0$), leading to an instability, $\gamma > 0$.

In the limit of $T_e \gg T_i$, since $v_d^2/2v_{th,e}^2 \to 0$:

$$\gamma \approx \sqrt{\frac{\pi}{8}} k v_d \left(\frac{m_e}{m_i}\right)^{1/2}$$

Exercise 2

Consider two electron populations with maxwellian distributions [1]:

$$f_p(v) = \frac{n_p}{\sqrt{2\pi} v_{th,p}} \exp\left(-\frac{v^2}{2\, v_{th,p}^2}\right) \quad \text{and} \quad f_b(v) = \frac{n_b}{\sqrt{2\pi} v_{th,b}} \exp\left(-\frac{(v - V)^2}{2\, v_{th,b}^2}\right) \tag{12}$$

where the population b has a drift velocity V, with $V \gg v_{th,b}$ (Figure 2).

[1] Consider 1-dimensional problem.

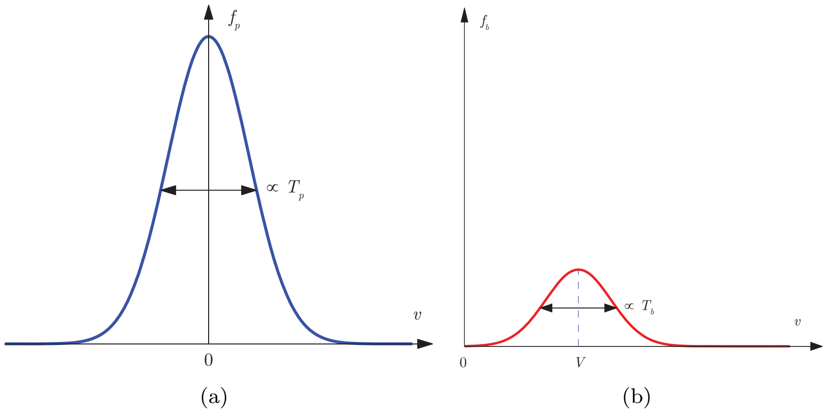

Figure 2: Distribution function of the two electron populations: (a) plasma and (b) beam.

Stability criteria

Let's define the total distribution function $F = f_p + f_b$ (figure 3). As we have seen during the lecture, the stability/instability of a perturbation with phase velocity v_ϕ is given by the sign of the derivative of F at $v = v_\phi$. When $dF/dv > 0$ the growth rate $\gamma \propto dF/dv$ is positive, therefore the amplitude of the initial perturbation will increase with time.

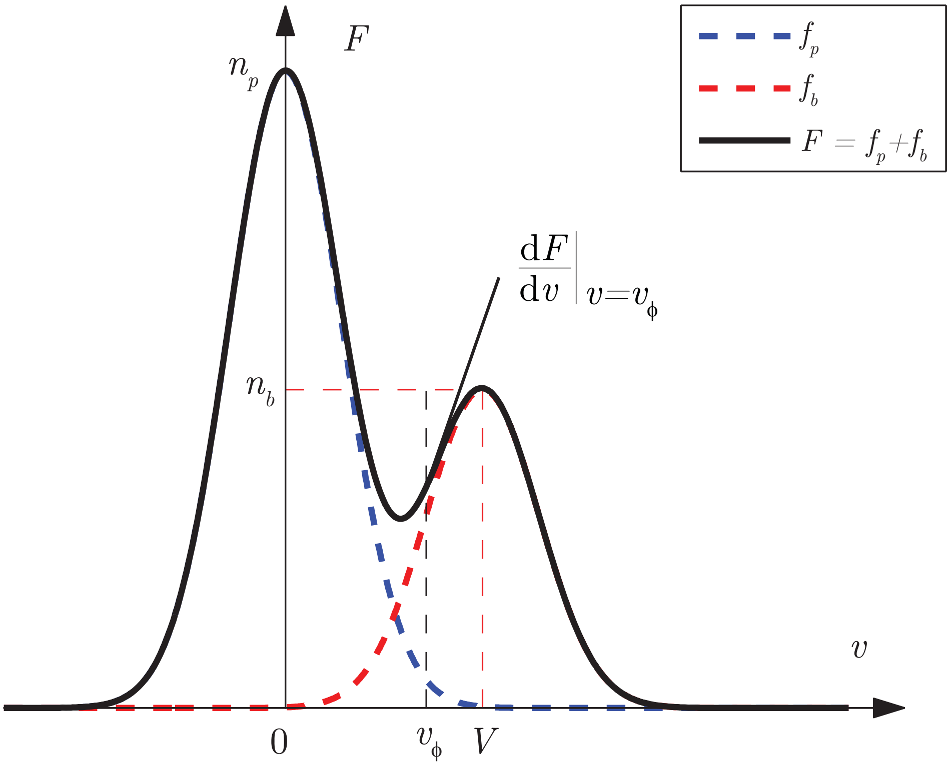

Figure 3: Total distribution function F. Notice that the slope of F for $v = v_\phi$ is related to the density ratio n_b/n_p.

The critical value of the slope of F below which no instability can occur is given by the condition (figure 4):

$$\left.\frac{dF}{dv}\right|_{v=v_\phi} = 0 \tag{13}$$

that corresponds to a critical density ratio $n_b^{\text{crit.}}/n_p$.

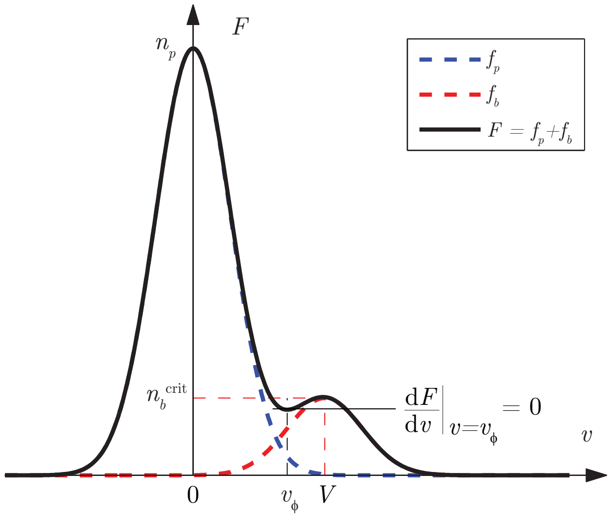

Figure 4: Critical value of dF/dv, after that $\gamma > 0$.

From the derivative of the total distribution function F:

$$\begin{aligned}\frac{dF}{dv} &= \frac{df_p}{dv} + \frac{df_b}{dv} = -\frac{v}{v_{th,p}^2} f_p(v) - \frac{v-V}{v_{th,b}^2} f_b(v) = \\ &= -\frac{n_p\, v}{\sqrt{2\pi} v_{th,p}^3}\, e^{-\frac{v^2}{2\, v_{th,p}^2}} - \frac{n_b^{\mathrm{crit.}}\, (v-V)}{\sqrt{2\pi} v_{th,b}^3}\, e^{-\frac{(v-V)^2}{2\, v_{th,b}^2}} \end{aligned} \tag{14}$$

Evaluation of the phase velocity v_ϕ

Suppose that the phase velocity corresponds to the value of v for which the slope of $f_b(v)$ is maximum. Since in the case of a maxwellian function the slope is maximum for $v_\phi \equiv v_{\mathrm{th}}$, if we have a drift velocity we obtain[2]:

$$v_\phi = V - v_{th,b} \tag{15}$$

To verify it:

$$\frac{df_b(v)}{dv} = -\frac{v-V}{v_{th,b}^2} f_b(v) \tag{16}$$

$$\frac{d^2 f_b(v)}{dv^2} = -\frac{1}{v_{th,b}^2} f_b(v) + \frac{(v-V)^2}{v_{th,b}^4} f_b(v) = \left[\frac{(v-V)^2}{v_{th,b}^2} - 1\right] \frac{f_b(v)}{v_{th,b}^2} \tag{17}$$

The maximum of df_b/dv is given by the equation:

$$\frac{d^2 f_b(v)}{dv^2} = 0 \Leftrightarrow \left[\frac{(v-V)^2}{v_{th,b}^2} - 1\right] = 0 \Rightarrow v = V \pm v_{th,b} \tag{18}$$

We can check in figure 2 that the solution $v = V - v_{th,b}$ corresponds the the maximum of f_b' and $v = V + v_{th,b}$ is the minimum[3].

[2] The solution $v_\phi = V + v_{th,b}$ is the minimum of df_b/dv.
[3] We can verify it from the sign of $d^3 f_b(v)/dv^3$.

Critical density value

Imposing now the condition in eq.(13), for $v = v_\phi = V - v_{th,b}$, we have:

$$\frac{dF}{dv}\Big|_{v=v_\phi} = -\frac{n_p(V - v_{th,b})}{\sqrt{2\pi}v_{th,p}^3}\, e^{-\frac{(V-v_{th,b})^2}{2\,v_{th,p}^2}} + \frac{n_b^{\text{crit.}}}{\sqrt{2\pi}v_{th,b}^2}\, e^{-\frac{1}{2}} = 0 \tag{19}$$

Finally, with the condition $V \gg v_{th,b}$, we find:

$$\frac{n_p\, V}{\sqrt{2\pi}v_{th,p}^3}\, e^{-\frac{V^2}{2\,v_{th,p}^2}} = \frac{n_b^{\text{crit.}}}{\sqrt{2\pi}v_{th,b}^2}\, e^{-\frac{1}{2}} \tag{20}$$

that gives a critical density ratio $n_b^{\text{crit.}}/n_p$ of:

$$\frac{n_b^{\text{crit.}}}{n_p} = \sqrt{e}\,\frac{v_{th,b}^2}{v_{th,p}^3}\, V\, e^{-\frac{V^2}{2\,v_{th,p}^2}} \tag{21}$$

and, since $v_{th,b}^2/v_{th,p}^2 \equiv T_b/T_p$,

$$\frac{n_b^{\text{crit.}}}{n_p} = \sqrt{e}\,\frac{T_b}{T_p}\,\frac{V}{v_{th,p}}\, e^{-\frac{V^2}{2\,v_{th,p}^2}} \tag{22}$$