

Plasma Physics I

Solution to the Series 11 (November 30, 2024)

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Exercise 1

- a) For the electron plasma waves (Langmuir waves), we have $|\omega/k| \gg v_{th,e}$. Therefore we can find the dispersion relation neglecting the contribution from the ion population. To calculate the damping rate of the wave:

$$\gamma = -\frac{\epsilon_i(\omega_r)}{\partial \epsilon_r / \partial \omega_r},$$

we need to evaluate the real part, $\epsilon_r(\omega_r)$, and the imaginary part, $\epsilon_i(\omega_r)$, of $\epsilon(\omega_r, k)$:

$$\begin{cases} \epsilon_r(\omega_r, k) = 1 - \frac{\omega_{pe}^2}{n_e k^2} \text{P.V.} \int du \frac{\frac{dF_{e0}}{du}}{u - \frac{\omega_r}{k}} \\ \epsilon_i(\omega_r, k) = -\pi \frac{\omega_{pe}^2}{n_e k^2} \frac{dF_{e0}}{du} \Big|_{u=\frac{\omega_r}{k}} \end{cases} \quad (1)$$

We have already solved the dispersion relation $\epsilon_r(\omega_r) = 0$ for the Langmuir waves (section 8.2, lecture IX), keeping the first three orders of the expansion in ku/ω :

$$\epsilon_r(\omega_r) \simeq 1 - \frac{\omega_{pe}^2}{\omega_r^2} \left\{ 1 + 3 \frac{k^2 v_{th,e}^2}{\omega_r^2} \right\} = 0.$$

For $\omega_r^2 \sim \omega_{pe}^2$, we have found the solution $\omega_r^2 \simeq \omega_{pe}^2 + 3k^2 v_{th,e}^2$.

To calculate the derivative $\partial \epsilon_r / \partial \omega_r$, we keep only the dominant term of the expansion (zero-order):

$$\frac{\partial \epsilon_r}{\partial \omega_r} = \frac{\partial}{\partial \omega_r} \left\{ 1 - \frac{\omega_{pe}^2}{\omega_r^2} \right\} = 2 \frac{\omega_{pe}^2}{\omega_r^3}. \quad (2)$$

Considering a Maxwellian distribution for the electrons:

$$F_{e0} = \frac{n_e}{\sqrt{2\pi} v_{th,e}} \exp \left\{ \left(-\frac{u^2}{2 v_{th,e}^2} \right) \right\}$$

with $v_{th,e}^2 = T_e/m_e$, we obtain:

$$\epsilon_i(\omega_r) = \sqrt{\frac{\pi}{2}} \frac{\omega_r \omega_{pe}^2}{k^3 v_{th,e}^3} \exp \left\{ -\frac{\omega_r^2}{2 k^2 v_{th,e}^2} \right\}. \quad (3)$$

The damping rate is therefore given by:

$$\gamma = -\sqrt{\frac{\pi}{8}} \frac{\omega_r^4}{k^3 v_{th,e}^3} \exp \left\{ -\frac{\omega_r^2}{2 k^2 v_{th,e}^2} \right\}. \quad (4)$$

We need now to substitute the expression for ω_r . Keeping only the dominant term $\omega \simeq \omega_{pe}$ and considering the correction in the exponential introduced by the third order term $\omega_r^2 \simeq \omega_{pe}^2 + 3k^2 v_{th,e}^2$ ¹, we find:

$$\gamma = -\sqrt{\frac{\pi}{8}} e^{-3/2} \frac{\omega_{pe}^4}{k^3 v_{th,e}^3} \exp \left\{ -\frac{\omega_{pe}^2}{2 k^2 v_{th,e}^2} \right\}. \quad (5)$$

b) Using the identity $\omega_{pe}/v_{th,e} \equiv \lambda_D^{-1}$, we can write:

$$\gamma = -\sqrt{\frac{\pi}{8}} e^{-3/2} \frac{\omega_{pe}}{(k\lambda_D)^3} \exp \left\{ -\frac{1}{2(k\lambda_D)^2} \right\}. \quad (6)$$

The important quantity is therefore the ratio between the Debye length and the wave length. It's easy to verify that $\gamma \rightarrow 0$ if $k\lambda_D \rightarrow 0$ and that there is a strong damping when $k\lambda_D \sim 1$. Imposing $d\gamma/d(k\lambda_D) = 0$, we can find the condition necessary to have the maximum damping rate $\gamma_{\max}$, that is $k\lambda_D = \sqrt{3}/3$.

c) From part b) we find that:

$$\left| \frac{\gamma}{\omega_r} \right| \simeq \left| \frac{\gamma}{\omega_{pe}} \right| = \left| -\sqrt{\frac{\pi}{8}} e^{-3/2} \frac{1}{(k\lambda_D)^3} \exp \left\{ -\frac{1}{2(k\lambda_D)^2} \right\} \right| < \left| \frac{\gamma_{\max}}{\omega_{pe}} \right| \simeq 0.16.$$

We thus indeed find that $\gamma/\omega_r \ll 1$

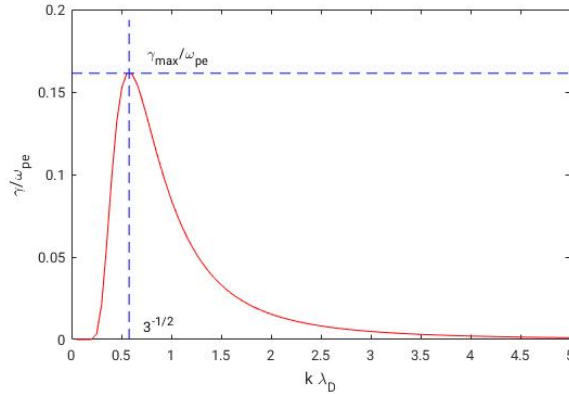


Figure 1: **Damping rate of a Langmuir wave:** plot of the function $f(x) = \frac{\gamma}{\omega_{pe}} = \sqrt{\pi/8} e^{-3/2} \frac{1}{x^3} \exp \left\{ \left(-\frac{1}{2x^2} \right) \right\}$.

¹The term $3k^2 v_{th,e}^2$ gives a factor $e^{-3/2} \approx 0.2$.

Exercise 2

Considering the equation:

$$\epsilon(\omega, k) = 1 + \sum_{\alpha} \frac{e^2}{m_{\alpha} \epsilon_0 k} \int_{\mathcal{L}} \frac{dF_{0,\alpha}}{du} \frac{1}{\omega - ku} du = 0 \quad (7)$$

for a hydrogen neutral plasma ($n_e = n_i = n_0$, $Z = 1$), with the approximation

$$kv_{th,i} \ll \omega \ll kv_{th,e} \quad T_e \gg T_i, \quad (8)$$

it is possible to derive the dispersion relation for the ion-acoustic waves (serie 9, exercise 2).

From the Landau's theory, the damping rate γ is given by:

$$\gamma \equiv -\frac{\epsilon_i(\omega_r, k)}{\partial \epsilon_r(\omega_r, k) / \partial \omega_r}, \quad (9)$$

where we can identify $\epsilon(\omega, k) = \epsilon_r(\omega, k) + i\epsilon_i(\omega, k) \equiv D(\omega, k)$ (electrostatic approximation) and the terms ω_r and γ , that are respectively the real part and the imaginary part of frequency ω :

$$\omega = \omega_r + i\gamma. \quad (10)$$

The eq.(9) is valid only if the absolute value of the damping rate is much smaller then the real part of the frequency ($|\gamma| \ll |\omega_r|$).

Real part of $\epsilon(\omega_r, k)$

The real part of $\epsilon(\omega_r, k)$ is given by:

$$\epsilon_r(\omega_r, k) = 1 + \sum_{\alpha} \frac{e^2}{m_{\alpha} \epsilon_0 k} \text{P.V.} \int_{-\infty}^{\infty} \frac{dF_{0,\alpha}}{du} \frac{1}{\omega_r - ku} du \quad (11)$$

Using directly the dispersion relation for the ion-acoustic waves, that we have found for $\omega \simeq \omega_r$ in serie 9 exercise 2, we have:

$$\begin{aligned} \epsilon_r(\omega_r, k) &\simeq 1 - \frac{e^2 n_i}{m_i \epsilon_0} \frac{1}{\omega_r^2} + \frac{e^2 n_e}{m_e \epsilon_0} \frac{1}{k^2 v_{th,e}^2} \\ &= 1 - \frac{\omega_{pi}^2}{\omega_r^2} + \frac{\omega_{pe}^2}{k^2 v_{th,e}^2}, \end{aligned} \quad (12)$$

that gives, assuming $\lambda \gg \lambda_D$:

$$\frac{\omega_r}{k} \simeq c_s \equiv \sqrt{\frac{T_e}{m_i}}, \quad (13)$$

where c_s is the ion-acoustic speed. The derivative of ϵ_r with respect to ω_r is therefore:

$$\frac{\partial \epsilon_r(\omega_r, k)}{\partial \omega_r} = 2 \frac{\omega_{pi}^2}{\omega_r^3} \quad (14)$$

Imaginary part of $\epsilon(\omega_r, k)$

The imaginary part of ϵ is given by:

$$\epsilon_i(\omega_r, k) = -\pi \sum_{\alpha} \frac{e^2}{m_{\alpha} \epsilon_0 k^2} \left. \frac{dF_{0,\alpha}}{du} \right|_{u=\omega_r/k}. \quad (15)$$

For a Maxwellian distribution function, we have:

$$\frac{dF_{0,\alpha}}{du} = -\frac{u}{v_{th,\alpha}^2} F_{0,\alpha}, \quad (16)$$

therefore

$$\epsilon_i(\omega_r, k) = \sqrt{\frac{\pi}{2}} \frac{e^2 n_0 \omega_r}{\epsilon_0 k^3} \left[\frac{\exp\left(-\frac{\omega_r^2}{2k^2 v_{th,e}^2}\right)}{m_e v_{th,e}^3} + \frac{\exp\left(-\frac{\omega_r^2}{2k^2 v_{th,i}^2}\right)}{m_i v_{th,i}^3} \right]. \quad (17)$$

Having $\omega_r/k = c_s \ll v_{th,e}$, we find:

$$\exp\left(-\frac{\omega_r^2}{2k^2 v_{th,e}^2}\right) \simeq 1. \quad (18)$$

Introducing the definition of electron and ion plasma frequency:

$$\omega_{p,e/i}^2 = \frac{e^2 n_0}{\epsilon_0 m_{e/i}} \quad (19)$$

and with the substitution $\omega_r/k = c_s$, we can rewrite ϵ_i as:

$$\epsilon_i(\omega_r, k) = \sqrt{\frac{\pi}{2}} \frac{c_s}{k^2} \left[\frac{\omega_{pe}^2}{v_{th,e}^3} + \frac{\omega_{pi}^2}{v_{th,i}^3} \exp\left(-\frac{c_s^2}{2v_{th,i}^2}\right) \right]. \quad (20)$$

Damping rate

From the results in the previous sections, eq.(14) and eq.(20), we can evaluate the damping rate using eq.(9):

$$\gamma = -\sqrt{\frac{\pi}{8}} \left[\frac{\omega_{pe}^2 c_s^4}{k^2 v_{th,e}^3} \frac{k^3}{\omega_{pi}^2} + \frac{\omega_{pi}^2 c_s^4}{k^2 v_{th,i}^3} \frac{k^3}{\omega_{pi}^2} \exp\left(-\frac{c_s^2}{2v_{th,i}^2}\right) \right]. \quad (21)$$

A possible simplification of the expression for γ is obtained introducing explicitly the thermal velocities and the sound speed. The result is:

$$\gamma = \underbrace{-\sqrt{\frac{\pi}{8}} k c_s \sqrt{\frac{m_e}{m_i}}}_{\gamma_e} - \underbrace{\sqrt{\frac{\pi}{8}} k c_s \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{T_e}{2T_i}\right)}_{\gamma_i} \quad (22)$$