Plasma Physics I

Solution to the Series 8 (November 9, 2024)

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Exercise 1

Residue theorem

Suppose f is a function of a complex variable z in a domain bound by a circle with radius C centered at a. Suppose f is an analytical function everywhere except at the point a. Therefore the function f has a pole at a and it can be represented as a Laurent series:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + ...$$

The coefficients of the series are given by:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

and, in particular, we have:

$$\oint_C f(z)\, dz = 2\pi i\, a_{-1}.$$

If no poles are enclosed by the integration path, the integral vanishes. We can evaluate the residue a_{-1} using the following expressions:

- First order pole:
$$a_{-1} = \lim_{z\to a}(z-a)f(z) \tag{1}$$

- k-th order pole:
$$a_{-1} = \lim_{z\to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}(z-a)^k f(z) \tag{2}$$

We can easily extend this approach to a function $f(z)$ integrated along an integration path Γ enclosing different poles situated at the points a, b, ... :

$$\oint_\Gamma f(z)dz = 2\pi i(a_{-1} + b_{-1} + ...) = 2\pi i \sum \text{Residues} \tag{3}$$

Lorentz distribution

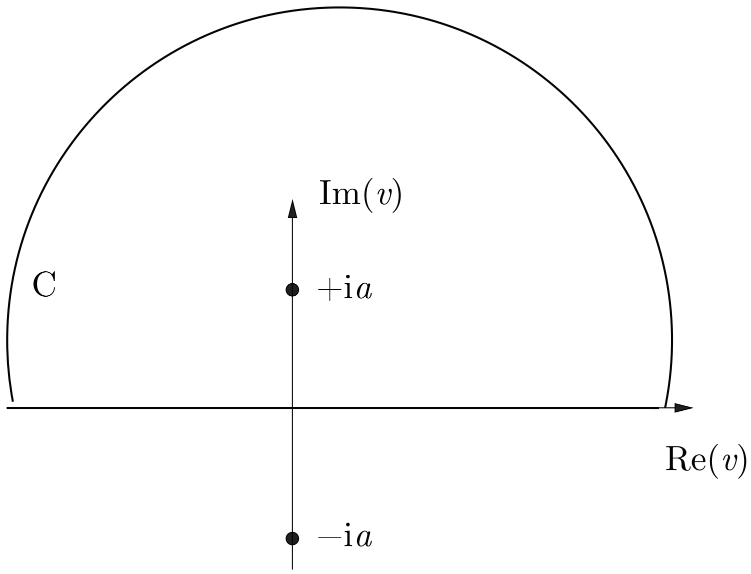

Figure 1: Integration path.

This function has two first order poles situated at $v = \mathrm{i}a$ and $v = -\mathrm{i}a$. We want to integrate it along the real axis, but we need to close somehow the integration path to use the residue theorem. We decide to do it with a semi-circle with radius $r \to \inf$, therefore the contribution to the integral is negligible (figure 1).

Only the pole at $v = +ia$ contributes to the integral. We evaluate the residue using the eq.(1):

$$a_{-1} = \lim_{v \to \mathrm{i}a} (v - ia) \frac{A}{a^2 + v^2} = \lim_{v \to \mathrm{i}a} (v - ia) \frac{A}{(v - \imath a)(v + ia)} = \frac{A}{2ia}$$

a) Using the residue theorem we can easily find the normalizing constant:

$$A = \frac{an}{\pi} \Rightarrow f(v) = \frac{n}{\pi} \frac{a}{v^2 + a^2} \tag{4}$$

b) The first moment of the distribution function is the mean velocity, that is zero due to the symmetry of $f(v)$:

$$\int_{-\infty}^{+\infty} f(v)\, v\, dv = \langle v \rangle = 0 \tag{5}$$

c) Conversely, the second order momentum of the distribution function, that is the mean kinetic energy, diverges because:

$$\lim_{v \to \pm\infty} f(v)\, v^2 = \lim_{v \to \infty} \frac{an}{\pi} \frac{v^2}{v^2 + a^2} = \frac{an}{\pi}. \tag{6}$$

We can also see this by doing the actual integration

$$\int_{-\infty}^{+\infty} \frac{n}{\pi} a \frac{v^2}{v^2+a^2} dv = \int_{-\infty}^{+\infty} \frac{n}{\pi} a \frac{v^2+a^2-a^2}{v^2+a^2} dv = \int_{-\infty}^{+\infty} \frac{n}{\pi} a (1 - \frac{a^2}{v^2+a^2}) dv \tag{7}$$

Therefore the solution is not physical and the lorentz distribution can not describe a real electron population.

Exercise 2

For the distribution function $f = f(\vec{x}, \vec{v}, t)$, the *Vlasov* equation is:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \vec{a} \cdot \frac{\partial f}{\partial \vec{v}} = 0 \tag{8}$$

with $\vec{a} \equiv \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B})$. We know that $\vec{x}$ and $\vec{v}$ are independent variables, therefore:

$$\frac{\partial}{\partial \vec{x}} \cdot \vec{v} = \frac{\partial}{\partial \vec{v}} \cdot \vec{E} = \frac{\partial}{\partial \vec{v}} \cdot (\vec{v} \times \vec{B}) = 0. \tag{9}$$

We can rewrite the Vlasov equation (8) as follows:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{x}} \cdot (\vec{v} f) + \frac{\partial}{\partial \vec{v}} \cdot (\vec{a} f) = 0. \tag{10}$$

The entropy is defined as:

$$S(t) \equiv -\int d^3v \int d^3x \, f(\vec{x}, \vec{v}, t) \, \ln f(\vec{x}, \vec{v}, t) \tag{11}$$

where the integrals are computed over the entire phase space.
We can now use the following mathematical property:
If $g(y) = \int f(x, y) \, dx$
then

$$\frac{dg(y)}{dy} = \frac{d}{dy} \int f(x, y) \, dx = \int \frac{\partial}{\partial y} f(x, y) \, dx. \tag{12}$$

Applying this to our case:

$$\begin{aligned} \frac{dS}{dt} &= -\int d^3v \int d^3x \, \frac{\partial}{\partial t}(f \ln f) \\ &= -\int d^3v \int d^3x \, (1 + \ln f) \frac{\partial f}{\partial t} \\ \text{with Vlasov} \quad &= +\int d^3v \int d^3x \, (1 + \ln f) \left[\frac{\partial}{\partial \vec{x}} \cdot (\vec{v} f) + \frac{\partial}{\partial \vec{v}} \cdot (\vec{a} f) \right]. \end{aligned} \tag{13}$$

Having

$$\frac{\partial}{\partial \vec{x}} \cdot (\vec{v} f \ln f) = \ln f \frac{\partial}{\partial \vec{x}} \cdot (\vec{v} f) + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = (1 + \ln f) \frac{\partial}{\partial \vec{x}} \cdot (\vec{v} f) \tag{14}$$

and

$$\frac{\partial}{\partial \vec{v}} \cdot (\vec{a} f \ln f) = \ln f \frac{\partial}{\partial \vec{v}} \cdot (\vec{a} f) + \vec{a} \cdot \frac{\partial f}{\partial \vec{v}} = (1 + \ln f) \frac{\partial}{\partial \vec{v}} \cdot (\vec{a} f), \tag{15}$$

we obtain

$$\begin{aligned} \frac{dS}{dt} &= \int d^3v \int d^3x \left[(1 + \ln f) \frac{\partial}{\partial \vec{x}} \cdot (\vec{v} f) + (1 + \ln f) \frac{\partial}{\partial \vec{v}} \cdot (\vec{a} f) \right] \\ &= \int d^3v \int d^3x \left[\frac{\partial}{\partial \vec{x}} \cdot (\vec{v} f \ln f) + \frac{\partial}{\partial \vec{v}} \cdot (\vec{a} f \ln f) \right]. \end{aligned} \tag{16}$$

Therefore,

$$\frac{dS}{dt} = \int d^3v \left\{ \int d^3x \left[\frac{\partial}{\partial \vec{x}} \cdot (\vec{v} f \ln f) \right] \right\} + \int d^3x \left\{ \int d^3v \left[\frac{\partial}{\partial \vec{v}} \cdot (\vec{a} f \ln f) \right] \right\}. \tag{17}$$

Using Gauss's theorem we can now write:

$$\int d^3x \left[\frac{\partial}{\partial \vec{x}} \cdot (\vec{v} f \ln f) \right] = \int_\Sigma d\vec{\sigma}_x \cdot \vec{v} f \ln f = 0. \tag{18}$$

This surface integral Σ is computed on the boundary of the phase space. Supposing that $f \to 0$ quickly when $\vec{v}$, $\vec{x}$ and $\vec{a} \to \infty$, the integral vanishes.

Analogously, we have:

$$\int d^3v \left[\frac{\partial}{\partial \vec{v}} \cdot (\vec{a} f \ln f) \right] = \int_\Sigma d\vec{\sigma}_v \cdot \vec{a} f \ln f = 0. \tag{19}$$

Finally,

$$\frac{dS}{dt} = 0 \tag{20}$$

the entropy is conserved in a plasma described by the Vlasov equation.