In a famous 1957 article, which earned them the Nobel Prize in physics (theory of superconductivity), Bardeen, Cooper, and Schrieffer (BCS) considered the following pairing Hamiltonian:

$$H_{BCS} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \frac{|V|}{\Omega} \sum_{\mathbf{k}\neq\mathbf{p}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}$$

with $\xi_{\mathbf{k}} = \mathbf{k}^2/2m - \epsilon_F$. Here, $-|V|$ is the attractive coupling constant between electrons with opposite spins and momenta, and Ω is the volume. In order to calculate the ground state of H_{BCS}, they built the states made of pairs:

$$|\Psi_{BCS}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right) |0\rangle \tag{1}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are variational parameters. In the exercice, we assume them to be real numbers.

(A.) Show that

$$\langle \Psi_{BCS} | \Psi_{BCS} \rangle = \prod_{\mathbf{k}} (u^2_{\mathbf{k}} + v^2_{\mathbf{k}}).$$

Therefore, the parameters $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ must satisfy $u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = 1$ in order for $|\Psi_{BCS}\rangle$ to be normalised.

(B.) Determine the expectation value of H_{BCS} in the state $|\Psi_{BCS}\rangle$

$$\langle \Psi_{BCS} | H_{BCS} | \Psi_{BCS} \rangle = 2\sum_{\mathbf{k}} \xi_{\mathbf{k}} v^2_{\mathbf{k}} - \frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) \left(\sum_{\mathbf{p}} u_{\mathbf{p}} v_{\mathbf{p}} \right)$$

(C.) We want to calculate the parameters $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ which minimise the energy.

1. Show that the variational equation

   $$\frac{\partial \langle H \rangle}{\partial v_{\mathbf{p}}} = 0$$

   is equivalent to (remember that $u^2_{\mathbf{k}} = 1 - v^2_{\mathbf{k}}$):

   $$2\xi_{\mathbf{p}} u_{\mathbf{p}} v_{\mathbf{p}} = \Delta (u^2_{\mathbf{p}} - v^2_{\mathbf{p}}) \tag{2}$$

   where the gap Δ is defined as

   $$\Delta = \frac{|V|}{\Omega} \sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}$$

2. Find expressions of $u^2_{\mathbf{k}}$ and $v^2_{\mathbf{k}}$ for the solutions of Eq.(2) and $u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = 1$. Choose the solution corresponding to the Fermi sea for the system without interaction ($V = 0 \Rightarrow \Delta = 0$):

   $$v_{\mathbf{k}} = \begin{cases} 1 & \text{if} \quad k < k_F \\ 0 & \text{if} \quad k > k_F \end{cases}$$

Show that in this case

$$
\begin{aligned}
u_{\mathbf{k}}^2 &= \frac{1}{2}\left(1+\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \\
v_{\mathbf{k}}^2 &= \frac{1}{2}\left(1-\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)
\end{aligned}
$$

with $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$.

3. Using the definition of Δ, derive the gap equation

$$1 = \frac{|V|}{2\Omega} \sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}}.$$