

BCS Hamiltonian:

$$H_{BCS} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \frac{|V|}{\Omega} \sum_{\mathbf{k} \neq \mathbf{p}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \quad (1)$$

BCS variational function:

$$|\Psi_{BCS}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger \right) |0\rangle \quad (2)$$

(A.) How to determine $\langle \Psi_{BCS} | \Psi_{BCS} \rangle$. In noting that $[c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}^\dagger, c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger] = 0$ if $\mathbf{k} \neq \mathbf{p}$, we have

$$\begin{aligned} \langle \Psi_{BCS} | \Psi_{BCS} \rangle &= \langle 0 | \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) \prod_{\mathbf{p}} (u_{\mathbf{p}} + v_{\mathbf{p}} c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger) | 0 \rangle \\ &= \langle 0 | \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) | 0 \rangle \\ &= \langle 0 | \prod_{\mathbf{k}} \left(u_{\mathbf{k}}^2 + u_{\mathbf{k}} v_{\mathbf{k}} \underbrace{c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger}_{\rightarrow 0} + u_{\mathbf{k}} v_{\mathbf{k}} \underbrace{c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}}_{\rightarrow 0} + v_{\mathbf{k}}^2 c_{-\mathbf{k}\downarrow} \underbrace{c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow}}_{1 - c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow}} c_{-\mathbf{k}\downarrow}^\dagger \right) | 0 \rangle \\ &= \langle 0 | \prod_{\mathbf{k}} \left(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 \underbrace{c_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow}^\dagger}_{1 - c_{-\mathbf{k}\downarrow}^\dagger c_{-\mathbf{k}\downarrow}} + v_{\mathbf{k}}^2 \underbrace{c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\uparrow}}_{\rightarrow 0} \right) | 0 \rangle \\ &= \prod_{\mathbf{k}} (u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2) \end{aligned}$$

So by choosing $u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$, we have

$$\langle \Psi_{BCS} | \Psi_{BCS} \rangle = 1 \quad (3)$$

(B.) To determine $\langle H_{BCS} \rangle \equiv \langle \Psi_{BCS} | H_{BCS} | \Psi_{BCS} \rangle$, begin with the kinetic part:

$$\sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} \langle c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} \rangle \quad (4)$$

For $\sigma = \uparrow$

$$\begin{aligned}
 \langle c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} \rangle &= \langle 0 | \prod_{\mathbf{q}} (u_{\mathbf{q}} + v_{\mathbf{q}} c_{-\mathbf{q}\downarrow} c_{\mathbf{q}\uparrow}) c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} \prod_{\mathbf{p}} (u_{\mathbf{p}} + v_{\mathbf{p}} c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger) | 0 \rangle \\
 &= \langle 0 | \underbrace{\left(\prod_{\mathbf{p} \neq \mathbf{k}} (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) (u_{\mathbf{p}} + v_{\mathbf{p}} c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger) \right)}_{\prod_{\mathbf{p} \neq \mathbf{k}} (u_{\mathbf{p}}^2 + v_{\mathbf{p}}^2) = 1} \\
 &\quad \times (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) | 0 \rangle \\
 &= \langle 0 | (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) | 0 \rangle
 \end{aligned}$$

more

$$\begin{aligned}
 c_{\mathbf{k}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) | 0 \rangle &= v_{\mathbf{k}} c_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger | 0 \rangle \\
 &= v_{\mathbf{k}} (1 - c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow}) c_{-\mathbf{k}\downarrow}^\dagger | 0 \rangle \\
 &= v_{\mathbf{k}} c_{-\mathbf{k}\downarrow}^\dagger | 0 \rangle
 \end{aligned}$$

and

$$\begin{aligned}
 \langle 0 | (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c_{\mathbf{k}\uparrow}^\dagger &= \left(c_{\mathbf{k}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) | 0 \rangle \right)^\dagger \\
 &= v_{\mathbf{k}} \langle 0 | c_{-\mathbf{k}\downarrow}
 \end{aligned}$$

from where

$$\begin{aligned}
 \langle 0 | (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) | 0 \rangle &= v_{\mathbf{k}}^2 \langle 0 | c_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow}^\dagger | 0 \rangle \\
 &= v_{\mathbf{k}}^2
 \end{aligned}$$

Likewise for $\sigma = \downarrow$

$$\begin{aligned}
 \langle c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\downarrow} \rangle &= \langle 0 | \prod_{\mathbf{q}} (u_{\mathbf{q}} + v_{\mathbf{q}} c_{-\mathbf{q}\downarrow} c_{\mathbf{q}\uparrow}) c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\downarrow} \prod_{\mathbf{p}} (u_{\mathbf{p}} + v_{\mathbf{p}} c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger) | 0 \rangle \\
 &= \langle 0 | \underbrace{\left(\prod_{\mathbf{p} \neq -\mathbf{k}} (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) (u_{\mathbf{p}} + v_{\mathbf{p}} c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger) \right)}_{\prod_{\mathbf{p} \neq -\mathbf{k}} (u_{\mathbf{p}}^2 + v_{\mathbf{p}}^2) = 1} \\
 &\quad \times (u_{-\mathbf{k}} + v_{-\mathbf{k}} c_{\mathbf{k}\downarrow} c_{-\mathbf{k}\uparrow}) c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\downarrow} (u_{-\mathbf{k}} + v_{-\mathbf{k}} c_{-\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\downarrow}^\dagger) | 0 \rangle \\
 &\quad \vdots \\
 &= v_{-\mathbf{k}}^2
 \end{aligned}$$

Finally, noticing that $\xi_{-\mathbf{k}} = \xi_{\mathbf{k}}$,

$$\sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} \langle c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} \rangle = 2 \sum_{\mathbf{k}} \xi_{\mathbf{k}} v_{\mathbf{k}}^2 \quad (5)$$

Let's now consider the interaction term:

$$-\frac{|V|}{\Omega} \sum_{\mathbf{k} \neq \mathbf{p}} \langle c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \rangle \quad (6)$$

We have (for $\mathbf{p} \neq \mathbf{k}$)

$$\begin{aligned}
\langle c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \rangle &= \langle 0 | \prod_{\mathbf{q}} (u_{\mathbf{q}} + v_{\mathbf{q}} c_{-\mathbf{q}\downarrow} c_{\mathbf{q}\uparrow}) c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \prod_{\mathbf{r}} (u_{\mathbf{r}} + v_{\mathbf{r}} c_{\mathbf{r}\uparrow}^\dagger c_{-\mathbf{r}\downarrow}^\dagger) | 0 \rangle \\
&= \langle 0 | \underbrace{\left(\prod_{\mathbf{q} \neq \mathbf{k}, \mathbf{p}} (u_{\mathbf{q}} + v_{\mathbf{q}} c_{-\mathbf{q}\downarrow} c_{\mathbf{q}\uparrow}) (u_{\mathbf{q}} + v_{\mathbf{q}} c_{\mathbf{q}\uparrow}^\dagger c_{-\mathbf{q}\downarrow}^\dagger) \right)}_{\prod_{\mathbf{q} \neq \mathbf{k}, \mathbf{p}} (u_{\mathbf{q}}^2 + v_{\mathbf{q}}^2) = 1} \\
&\quad \times (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger \\
&\quad \times c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) (u_{\mathbf{p}} + v_{\mathbf{p}} c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger) | 0 \rangle \\
&= \langle 0 | (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger \\
&\quad \times c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) (u_{\mathbf{p}} + v_{\mathbf{p}} c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger) | 0 \rangle
\end{aligned}$$

More

$$c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) (u_{\mathbf{p}} + v_{\mathbf{p}} c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger) | 0 \rangle = v_{\mathbf{p}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) | 0 \rangle \quad (7)$$

and

$$\langle 0 | (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger = \langle 0 | v_{\mathbf{k}} (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) \quad (8)$$

from where

$$\begin{aligned}
\langle c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \rangle &= \langle 0 | v_{\mathbf{k}} (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) v_{\mathbf{p}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger) | 0 \rangle \\
&= u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{p}} v_{\mathbf{p}}
\end{aligned}$$

and

$$\begin{aligned}
-\frac{|V|}{\Omega} \sum_{\mathbf{k} \neq \mathbf{p}} \langle c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \rangle &= -\frac{|V|}{\Omega} \sum_{\mathbf{k} \neq \mathbf{p}} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{p}} v_{\mathbf{p}} \\
&\simeq -\frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) \left(\sum_{\mathbf{p}} u_{\mathbf{p}} v_{\mathbf{p}} \right)
\end{aligned}$$

So we finally get:

$$\langle \Psi_{BCS} | H_{BCS} | \Psi_{BCS} \rangle = 2 \sum_{\mathbf{k}} \xi_{\mathbf{k}} v_{\mathbf{k}}^2 - \frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) \left(\sum_{\mathbf{p}} u_{\mathbf{p}} v_{\mathbf{p}} \right) \quad (9)$$

(C.)

1. We have

$$\begin{aligned}
\frac{\partial \langle H_{BCS} \rangle}{\partial v_{\mathbf{p}}} &= 2 \sum_{\mathbf{k}} \xi_{\mathbf{k}} \frac{\partial}{\partial v_{\mathbf{p}}} v_{\mathbf{k}}^2 - \frac{|V|}{\Omega} \frac{\partial}{\partial v_{\mathbf{p}}} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right)^2 \\
&= 4 \sum_{\mathbf{k}} \xi_{\mathbf{k}} v_{\mathbf{k}} \frac{\partial v_{\mathbf{k}}}{\partial v_{\mathbf{p}}} - 2 \frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) \left(\sum_{\mathbf{k}} \frac{\partial u_{\mathbf{k}}}{\partial v_{\mathbf{p}}} v_{\mathbf{k}} + \frac{\partial v_{\mathbf{k}}}{\partial v_{\mathbf{p}}} u_{\mathbf{k}} \right)
\end{aligned}$$

More

$$\frac{\partial v_{\mathbf{k}}}{\partial v_{\mathbf{p}}} = \delta_{\mathbf{k}, \mathbf{p}} \quad (10)$$

and using $u_{\mathbf{k}}^2 = 1 - v_{\mathbf{k}}^2$

$$\begin{aligned}\frac{\partial u_{\mathbf{k}}}{\partial v_{\mathbf{p}}} &= \frac{\partial \sqrt{1 - v_{\mathbf{k}}^2}}{\partial v_{\mathbf{p}}} \\ &= -\frac{v_{\mathbf{k}}}{\sqrt{1 - v_{\mathbf{k}}^2}} \delta_{k,p} \\ &= -\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} \delta_{k,p}\end{aligned}$$

so we have

$$\frac{\partial \langle H_{BCS} \rangle}{\partial v_{\mathbf{p}}} = 4\xi_{\mathbf{p}} v_{\mathbf{p}} - 2 \frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) \left(-\frac{v_{\mathbf{p}}^2}{u_{\mathbf{p}}} + u_{\mathbf{p}} \right) \quad (11)$$

and

$$\begin{aligned}\frac{\partial \langle H_{BCS} \rangle}{\partial v_{\mathbf{p}}} &= 0 \\ \Leftrightarrow \\ 2\xi_{\mathbf{p}} v_{\mathbf{p}} u_{\mathbf{p}} &= \frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) (u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2)\end{aligned} \quad (12)$$

By defining

$$\Delta = \frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) \quad (13)$$

Eq.(12) becomes

$$2\xi_{\mathbf{p}} v_{\mathbf{p}} u_{\mathbf{p}} = \Delta (u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2) \quad (14)$$

2. We have the 2 equations

$$2\xi_{\mathbf{p}} v_{\mathbf{p}} u_{\mathbf{p}} = \Delta (u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2) \quad (15)$$

and

$$u_{\mathbf{p}}^2 + v_{\mathbf{p}}^2 = 1 \quad (16)$$

to solve for the 2 unknowns $(u_{\mathbf{p}}, v_{\mathbf{p}})$. Of Eq.(16) we get $u_{\mathbf{p}} = \sqrt{1 - v_{\mathbf{p}}^2}$ that we use in Eq.(15). We obtain:

$$\begin{aligned}\frac{2\xi_{\mathbf{p}}}{\Delta} \sqrt{1 - v_{\mathbf{p}}^2} v_{\mathbf{p}} &= (1 - 2v_{\mathbf{p}}^2) \\ \Rightarrow \\ \frac{4\xi_{\mathbf{p}}^2}{\Delta^2} (1 - v_{\mathbf{p}}^2) v_{\mathbf{p}}^2 &= (1 - 2v_{\mathbf{p}}^2)^2 \\ \Leftrightarrow \\ 4 \frac{\Delta^2 + \xi_{\mathbf{p}}^2}{\Delta^2} v_{\mathbf{p}}^4 - 4 \frac{\Delta^2 + \xi_{\mathbf{p}}^2}{\Delta^2} v_{\mathbf{p}}^2 + 1 &= 0\end{aligned}$$

We put $E_{\mathbf{p}}^2 = \Delta^2 + \xi_{\mathbf{p}}^2$ and we solve in $v_{\mathbf{p}}^2$:

$$\begin{aligned}v_{\mathbf{p}}^2 &= \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{\Delta^2}{E_{\mathbf{p}}^2}} \right) \\ &= \frac{1}{2} \left(1 \pm \frac{|\xi_{\mathbf{p}}|}{E_{\mathbf{p}}} \right)\end{aligned}$$

Choose the solution for Fermi Sea for the free system ($V = 0 \Rightarrow \Delta = 0$):

$$v_{\mathbf{p}} = \begin{cases} 1 & \text{if } p < k_F \\ 0 & \text{if } p > k_F \end{cases} \quad (17)$$

So you have to choose

$$v_{\mathbf{p}}^2 = \begin{cases} \frac{1}{2} \left(1 + \frac{|\xi_{\mathbf{p}}|}{E_{\mathbf{p}}} \right) & \text{if } p < k_F \\ \frac{1}{2} \left(1 - \frac{|\xi_{\mathbf{p}}|}{E_{\mathbf{p}}} \right) & \text{if } p > k_F \end{cases} \quad (18)$$

Noticing that $\xi_{\mathbf{p}} < 0$ when $p < k_F$ and $\xi_{\mathbf{p}} > 0$ when $p > k_F$, this result can be summarized

$$v_{\mathbf{p}}^2 = \frac{1}{2} \left(1 - \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}} \right) \quad (19)$$

By injecting Eq.(19) into Eq.(16), we obtain

$$\begin{aligned} u_{\mathbf{p}}^2 &= 1 - v_{\mathbf{p}}^2 \\ &= \frac{1}{2} \left(1 + \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}} \right) \end{aligned}$$

In summary

$$\begin{cases} u_{\mathbf{p}}^2 = \frac{1}{2} \left(1 + \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}} \right) \\ v_{\mathbf{p}}^2 = \frac{1}{2} \left(1 - \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}} \right) \\ E_{\mathbf{p}}^2 = \Delta^2 + \xi_{\mathbf{p}}^2 \end{cases} \quad (20)$$

3. Let's go back to the definition of the gap Eq.(13) with the results Eq.(20) and $E_{\mathbf{k}}^2 = \Delta^2 + \xi_{\mathbf{k}}^2$:

$$\begin{aligned} \Delta &= \frac{|V|}{\Omega} \sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \\ &= \frac{|V|}{2\Omega} \sum_{\mathbf{k}} \sqrt{\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \right) \left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \right)} \\ &= \frac{|V|}{2\Omega} \sum_{\mathbf{k}} \sqrt{\left(1 - \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^2} \right)} \\ &= \frac{\Delta |V|}{2\Omega} \sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}} \end{aligned}$$

In addition to the trivial solution $\Delta = 0$ which corresponds to the normal state, this equation has a solution $\Delta \neq 0$ (superconducting state), which is solution of the equation of the gap:

$$1 = \frac{|V|}{2\Omega} \sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}} \quad (21)$$