The Schrödinger equation is given by

$$\begin{cases} H = -\frac{1}{2m}\nabla_1^2 - \frac{1}{2m}\nabla_2^2 + V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) \\ H\Psi = E\Psi \end{cases} \tag{1}$$

For two electrons with total momentum $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2 = 0$, $\boldsymbol{k}_1 = -\boldsymbol{k}_2 = \boldsymbol{k}$, the wave function of the relative motion can be decomposed into plane waves as

$$\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \Psi(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)}. \tag{2}$$

We also set $\hbar = 1$.

1. With $\epsilon_{\boldsymbol{k}} = \boldsymbol{k}^2/(2m)$ and using

$$\begin{cases} \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \\ \boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2) \end{cases} \tag{3}$$

we have

$$\begin{aligned} H\Psi(\boldsymbol{r}) &= \left(-\frac{1}{2m}\nabla_1^2 - \frac{1}{2m}\nabla_2^2 + V(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right) \sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)} \\ &= \sum_{\boldsymbol{k}} \left(\frac{\boldsymbol{k}^2}{2m} + \frac{\boldsymbol{k}^2}{2m} + V(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right) \Psi_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)} \\ &= \sum_{\boldsymbol{k}} \left(2\epsilon_{\boldsymbol{k}} + V(\boldsymbol{r})\right) \Psi_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \end{aligned}$$

After a Fourier transform,

$$\begin{aligned} \int d^3\boldsymbol{r} H\Psi(\boldsymbol{r}) e^{-i\boldsymbol{k}'\boldsymbol{r}} &= \sum_{\boldsymbol{k}} 2\epsilon_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} \underbrace{\int d^3\boldsymbol{r} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}}_{=\Omega\delta_{\boldsymbol{k},\boldsymbol{k}'}} + \sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} \underbrace{\int d^3\boldsymbol{r} V(\boldsymbol{r}) e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}}_{\equiv V_{\boldsymbol{k}'-\boldsymbol{k}}} \\ &= \Omega 2\epsilon_{\boldsymbol{k}'} \Psi_{\boldsymbol{k}'} + \sum_{\boldsymbol{k}} V_{\boldsymbol{k}'-\boldsymbol{k}} \Psi_{\boldsymbol{k}} \end{aligned}$$

Similarly, for $E\Psi(\boldsymbol{r})$,

$$\int d^3\boldsymbol{r} E\Psi(\boldsymbol{r}) e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}} = E\sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} \underbrace{\int d^3\boldsymbol{r} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}}_{=\Omega\delta_{\boldsymbol{k},\boldsymbol{k}'}} = \Omega E \Psi_{\boldsymbol{k}'}$$

Thus, the Schrödinger equation in momentum space is given by

$$\left(E - 2\epsilon_{\boldsymbol{k}'}\right) \Psi_{\boldsymbol{k}'} = \frac{1}{\Omega} \sum_{\boldsymbol{k}} V_{\boldsymbol{k}'-\boldsymbol{k}} \Psi_{\boldsymbol{k}} \tag{4}$$

2. Let us consider the case $V_{\boldsymbol{k}'-\boldsymbol{k}} = -|v|$, then the equation (4) becomes

$$\Psi_{\boldsymbol{k}'} = -\frac{|v|}{\Omega}\frac{1}{E-2\epsilon_{\boldsymbol{k}'}}\sum_{\boldsymbol{k}}\Psi_{\boldsymbol{k}} \tag{5}$$

Summing over $\boldsymbol{k}'$ on both sides of the equation,

$$\sum_{\boldsymbol{k}'}\Psi_{\boldsymbol{k}'} = -\frac{|v|}{\Omega}\sum_{\boldsymbol{k}'}\frac{1}{E-2\epsilon_{\boldsymbol{k}'}}\sum_{\boldsymbol{k}}\Psi_{\boldsymbol{k}} \tag{6}$$

and thus

$$-\frac{1}{|v|} = \frac{1}{\Omega}\sum_{\boldsymbol{k}'}\frac{1}{E-2\epsilon_{\boldsymbol{k}'}} \tag{7}$$

3. We assume

$$\Psi_{\boldsymbol{k}} = 0 \quad \text{if} \quad |k| < k_F \tag{8}$$

and

$$V_{\boldsymbol{k}'-\boldsymbol{k}} = \begin{cases} -|v| & \text{if} \quad \epsilon_F < \epsilon_{\boldsymbol{k}}, \epsilon_{\boldsymbol{k}'} < \epsilon_F + \omega_D \\ 0 & \text{if not} \end{cases} \tag{9}$$

Thus, by defining k_0 by $k_0^2/2m = \epsilon_F + \omega_D = k_F^2/2m + \omega_D$, we obtain

$$\begin{aligned}
\frac{1}{\Omega}\sum_{\boldsymbol{k}'}\frac{1}{E-2\epsilon_{\boldsymbol{k}'}} &= \frac{1}{(2\pi)^3}\int\frac{d^3\boldsymbol{k}'}{E-2\epsilon_{\boldsymbol{k}'}} = \frac{4\pi m}{(2\pi)^3}\int_{k_F}^{k_0}\frac{k'^2}{mE-k'^2}dk' \\
&= \frac{4\pi m}{(2\pi)^3}\left[-k' + \frac{\sqrt{mE}}{2}\ln\left(\frac{k'+\sqrt{mE}}{k'-\sqrt{mE}}\right)\right]_{k_F}^{k_0} \\
&= \frac{4\pi m}{(2\pi)^3}\left(k_F - k_0 + \frac{\sqrt{mE}}{2}\ln\left(\frac{\left(k_0+\sqrt{mE}\right)\left(k_F-\sqrt{mE}\right)}{\left(k_0-\sqrt{mE}\right)\left(k_F+\sqrt{mE}\right)}\right)\right)
\end{aligned}$$

But for $\omega_D \ll \epsilon_F$ we have $k_0 = \sqrt{2m\epsilon_F(1+\frac{\omega_D}{\epsilon_F})} \simeq k_F + m\omega_D/k_F$.

On the other hand $E = 2\epsilon_F - \epsilon_b$ and with $\epsilon_b \ll \epsilon_F \Rightarrow \sqrt{mE} \simeq k_F - \frac{m\epsilon_b}{2k_F}$. The argument of the log therefore becomes

$$\begin{aligned}
\frac{\left(k_0+\sqrt{mE}\right)\left(k_F-\sqrt{mE}\right)}{\left(k_0-\sqrt{mE}\right)\left(k_F+\sqrt{mE}\right)} &\simeq \frac{\left(k_F+\frac{m\omega_D}{k_F}+k_F-\frac{m\epsilon_b}{2k_F}\right)\left(k_F-k_F+\frac{m\epsilon_b}{2k_F}\right)}{\left(k_F+\frac{m\omega_D}{k_F}-k_F+\frac{m\epsilon_b}{2k_F}\right)\left(k_F+k_F-\frac{m\epsilon_b}{2k_F}\right)} \\
&= \frac{\left(2k_F+\frac{m\omega_D}{k_F}-\frac{m\epsilon_b}{2k_F}\right)\frac{m\epsilon_b}{2k_F}}{\left(\frac{m\omega_D}{k_F}+\frac{m\epsilon_b}{2k_F}\right)\left(2k_F-\frac{m\epsilon_b}{2k_F}\right)} \\
&= \frac{m\epsilon_b + \frac{m^2\omega_D\epsilon_b}{2k_F^2} - \left(\frac{m\epsilon_b}{2k_F}\right)^2}{2m\omega_D + m\epsilon_b - \frac{m^2\omega_D\epsilon_b}{2k_F^2} - \left(\frac{m\epsilon_b}{2k_F}\right)^2} \\
&\simeq \frac{\epsilon_b}{2\omega_D+\epsilon_b}
\end{aligned}$$

and

$$\int\frac{d^3\boldsymbol{k}'}{E-2\epsilon_{\boldsymbol{k}'}} \simeq 2\pi m k_F \ln\left(\frac{\epsilon_b}{2\omega_D+\epsilon_b}\right) \tag{10}$$

Eq.(7) then becomes

$$-\frac{1}{|v|} \simeq \frac{mk_F}{(2\pi)^2} \ln\left(\frac{\epsilon_b}{2\omega_D + \epsilon_b}\right) \tag{11}$$

and we get

$$\epsilon_b \simeq 2\omega_D \left(e^{\frac{2}{\rho_F |v|}} - 1\right)^{-1} \tag{12}$$

with $\rho_F = mk_F/(2\pi^2)$ the density of states at the Fermi level.

This calculation could also be done in the following way:

$$\frac{1}{|v|} = \frac{1}{\Omega} \sum_{\boldsymbol{k'}} \frac{1}{2\epsilon_{\boldsymbol{k'}} - E} \quad = \quad \int_{\epsilon_F}^{\epsilon_F + \omega_D} d\epsilon \rho(\epsilon) \frac{1}{2\epsilon - E}$$

This integral is restricted to energies close to ϵ_F (at a maximum distance of the order of ω_D) for which $\rho(\epsilon) \simeq \rho_F$, and so

$$\begin{aligned} \frac{1}{|v|} &\simeq \rho_F \int_{\epsilon_F}^{\epsilon_F + \omega_D} d\epsilon \frac{1}{2\epsilon - E} = \frac{1}{2}\rho_F \ln\left(\frac{2(\epsilon_F + \omega_D) - E}{2\epsilon_F - E}\right) \\ &= \frac{1}{2}\rho_F \ln\left(1 + \frac{2\omega_D}{\epsilon_b}\right), \end{aligned}$$

which gives exactly the same result.

4. We immediately get the expression in the limit $\rho_F|v| \ll 1$,

$$\epsilon_b \simeq 2\omega_D e^{-\frac{2}{\rho_F |v|}}, \tag{13}$$

and in the limit $\rho_F|v| \gg 1$

$$\epsilon_b \simeq \omega_D \rho_F |v|. \tag{14}$$